TOPS

COMPOSITION BOOK

MAE 5730

DYNAMICS AND

VIBRATIONS

FALL 2013

Item No. 63796

College Rule • 100 Sheets • 9¾" x 7½"

8/28/13

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<u>Course Info</u>: ① More MATLAB

② More mechanisms

③ Less "physicsy" stuff

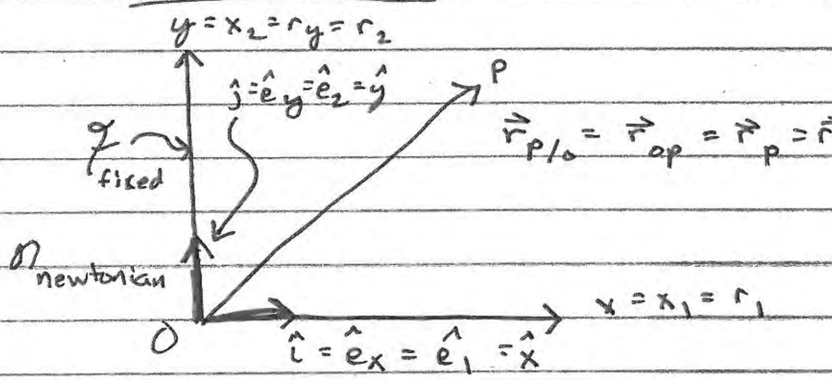
④ Less 3D

<u>Textbooks</u>: Taylor, Classical Mechanics

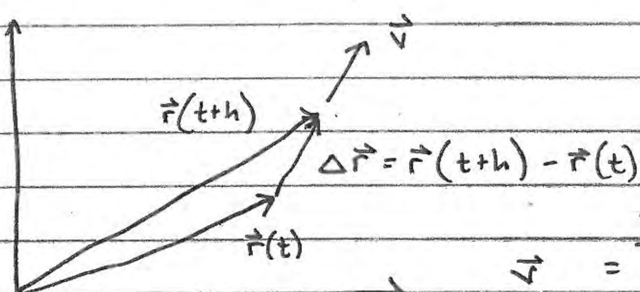
Tongue, Vibrations

<u>Vectors</u>:

<u>Position Vectors</u>



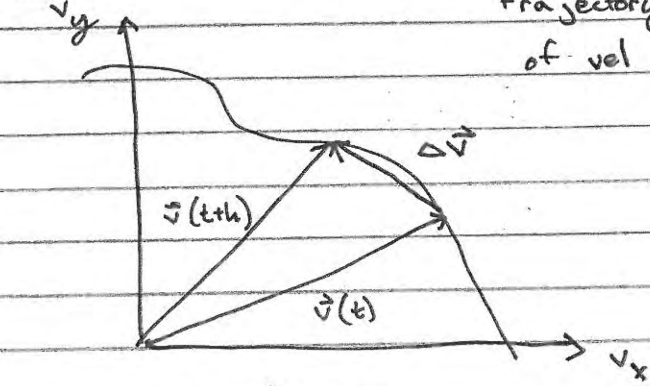
<u>Velocity</u>



$$\vec{v} = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \dot{\vec{r}} = \frac{d}{dt}\vec{r}$$

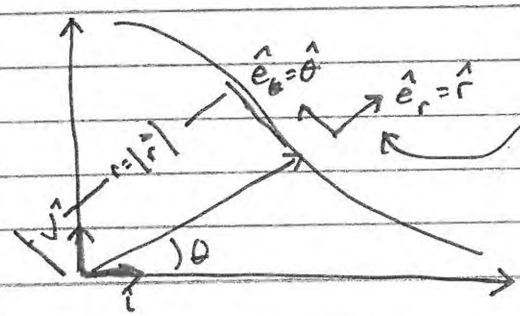
$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$$
$$= \hat{x}\dot{x} + \hat{y}\dot{y}$$

acceleration



trajectory of the tip of vel vector ⇒ "hodograph", Hamilton

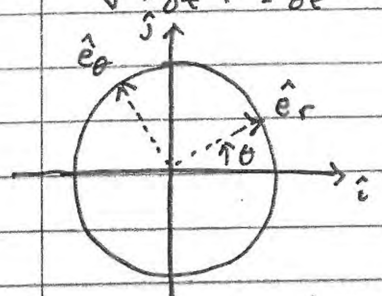
$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \dot{v}_x \hat{i} + \dot{v}_y \hat{j}$$
$$= \ddot{x}\hat{i} + \ddot{y}\hat{j}$$



time dependent base vectors

$$\hat{e}_r \equiv \frac{\vec{r}}{|\vec{r}|}$$

$$\vec{r} = r\hat{e}_r$$
$$\vec{v} = \frac{\partial}{\partial t}\vec{r} = \frac{\partial}{\partial t} r\hat{e}_r = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r$$



$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

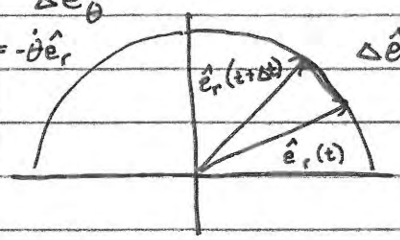
$$\hat{e}_\theta = \sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\dot{\hat{e}}_r = \dot{\theta}\,\hat{e}_\theta$$

$$\dot{\hat{e}}_\theta = -\dot{\theta}\,\hat{e}_r$$

$\Delta\hat{e}_\theta$

$\Rightarrow \dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$



$\Delta\hat{e}_r = (\Delta\theta)\,\hat{e}_\theta \Rightarrow \dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta$

$\vec{a}_r = \frac{\partial}{\partial t}\dot{r}\hat{e}_r$

$\vec{a}_r = \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r$

$\vec{a}_r = \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta$

$\vec{a}_\theta = \frac{\partial}{\partial t}(r\dot{\theta}\hat{e}_\theta)$

$\vec{a}_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

back to $\vec{v}$

$\vec{v} = \dot{r}\hat{e}_r + r\underbrace{\dot{\hat{e}}_r}_{\dot{\theta}\hat{e}_\theta}$

$= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

$\vec{a} = \frac{\partial}{\partial t}\vec{v} = \frac{\partial}{\partial t}\left[\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\right]$

$= \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta$

$= \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\hat{e}_r)$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta}$$

NOTE: $\vec{r} = \vec{r}$

$x\hat{\imath} + y\hat{\jmath} = r\hat{e}_r$

$\vec{v} = \vec{v}$

$\dot{x}\hat{\imath} + \dot{y}\hat{\jmath} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

$\vec{a} = \vec{a}$

$\ddot{x}\hat{\imath} + \ddot{y}j = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

8/30/13

Procedure

① Reframe (O, coordinate system)

② FBD

③ LMB (AMB)

$F = ma$

$\vec{F} = m \frac{\partial^2 \vec{r}}{\partial t^2}$

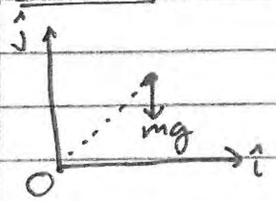
④ differential equation

↳ hand

↳ MATLAB

} energy methods

Ballistics



$\vec{F} = m\vec{a}$

$m \frac{\partial^2 \vec{r}}{\partial t^2} = -mg\hat{j}$ $\Rightarrow \ddot{x} = 0$

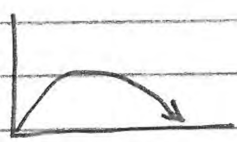
$\ddot{\vec{r}} = -g\hat{j}$ $\ddot{y} = -g$

$\dot{\vec{r}} = \vec{v}_0 - gt\hat{j}$

$\vec{r}_0 = 0$ $\vec{r} = \vec{r}_0 + \vec{v}_0 t - \frac{1}{2}t^2\hat{j}$

$\vec{v}_0 = v_0\cos\theta\,\hat{\imath} + v_0\sin\theta\,\hat{j}$

$\vec{r} = v_0\cos\theta\, t\,\hat{\imath} + \left(v_0\sin\theta\, t - \frac{1}{2}gt^2\right)\hat{j}$



$x = v_0\cos\theta\, t$

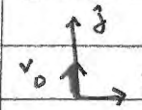
$y = v_0\sin\theta\, t - \frac{1}{2}gt^2$

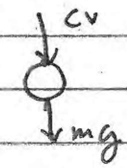
DRAG

F_{linear} $= cv$
viscous drag — opposite direction to velocity

$F_{quadratic}$ $= cv^2$ $= c_D \frac{1}{2}\rho A v^2$
drag — opposite direction to velocity

Ballistics (with linear drag)





$$\vec{F} = m\vec{a}$$

$$m\ddot{\vec{r}} = -mg\hat{j} - cv\hat{j}$$

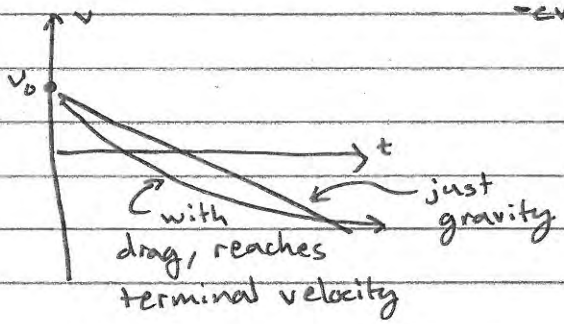
$$\ddot{\vec{r}} = -g\hat{j} - \frac{c}{m}v\hat{j}$$

$$\ddot{\vec{r}} = \left(-g - \frac{c}{m}\left|\dot{\vec{r}}\right|\right)\hat{j}$$

$$m\ddot{y} = -c\dot{y} - mg$$

$$\boxed{m\dot{v} = -cv - mg}$$

$$\frac{m\,dv}{-cv - mg} = dt$$



$$m\frac{\partial v}{\partial t} = -cv - mg$$

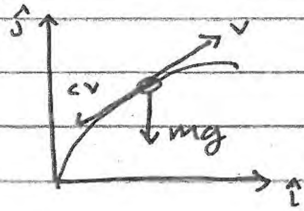
$$cv = -m\left(\frac{\partial v}{\partial t} + g\right)$$

$$v(t) = -v_t + (v_t - v_0)e^{-t/\tau}$$

$$\tau = m/c$$

Terminal velocity $\Rightarrow cv_T = -mg$

$$v_T = \frac{-mg}{c}$$



LMB

$$m\frac{\partial \vec{r}^2}{\partial t^2} = -mg\hat{j} - c\vec{v}$$

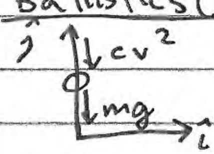
$$m(\ddot{x}\hat{i} + \ddot{y}\hat{j}) = -mg\hat{j} - c(\dot{x}\hat{i} + \dot{y}\hat{j})$$

$$m\ddot{x} = -c\dot{x}$$

$$m\ddot{y} = -c\dot{y} - mg$$

Ballistics (with Qudratic Drag)

1D



$$m\frac{\partial \vec{r}^2}{\partial t^2} = -mg\hat{j} - cv^2\hat{j}$$

$$m\ddot{y} = -c\dot{y}^2 - mg$$

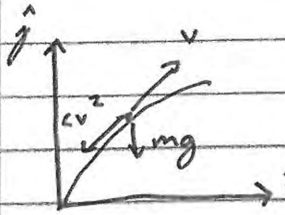
$$m\dot{v} = -cv^2 - mg \quad (v>0)$$

$$m\dot{v} = cv^2 - mg \quad (v<0)$$

terminal velocity:

$$\boxed{v_t = -\sqrt{\frac{mg}{c}}}$$

2D



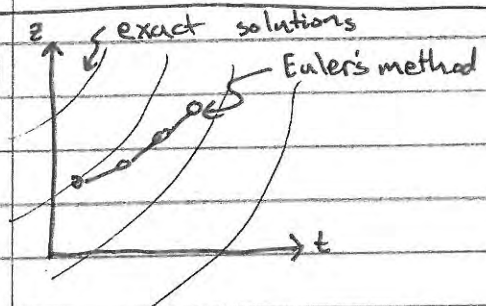
$$m\ddot{\vec{r}} = -mg\hat{j} - cv^2 \frac{\vec{v}}{|\vec{v}|}$$

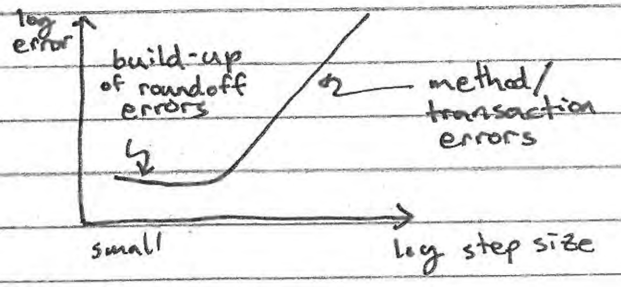
$$\boxed{m\ddot{\vec{r}} = -mg\hat{j} - cv\vec{v}}$$

$$m\ddot{y} = -mg - c\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$m\ddot{x} = -c\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}$$

9/4/13





Error in ODE soln

General Problem

$\vec{F} = \vec{F}(\vec{r}, \vec{v}, t)$

e.g. gravity

$$\vec{F} = \begin{cases} -mg\hat{j} & \text{near earth} \\ -\frac{MmG}{r^3}\vec{r} & \text{big } G \end{cases}$$

spring

$$\vec{F} = k\left(|\vec{r}| - l_0\right)\frac{\vec{r}}{|\vec{r}|}$$

drag

$$\vec{F} = \begin{cases} -c\vec{v} & \text{linear viscous} \\ -c|\vec{v}|\vec{v} & \text{quadratic viscous} \end{cases}$$

$\boxed{\vec{F} = m\vec{a}}$ $\quad \vec{a} = \vec{F}/m \quad$ know $\vec{F}(\vec{r}, \vec{v}, t)$

$\dot{\vec{r}} = v$ $\qquad$ $\dot{\vec{v}} = \vec{a} \leftarrow \vec{F}/m$

$\Rightarrow \dot{z} = f(z,t) \leftarrow$ calc of $\dot{\vec{r}}$ + $\dot{\vec{v}}$

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ v_x \\ v_y \end{bmatrix}$$

Ballistics Problem

$\vec{F} = -c|\vec{v}|\vec{v} - mg\hat{j}$

$\hookrightarrow c = c_D \rho_{air} A_{cross}$

LMB: linear momentum balance

$\vec{F} = m\vec{a}$

$-c|\vec{v}|\vec{v} - mg\hat{j} = m\vec{a}$

$\vec{a} = \frac{-c}{m}|\vec{v}|\vec{v} - g\hat{j}$

$$\left.\begin{aligned} &\dot{\vec{r}} = \vec{v} \\ &\dot{\vec{v}} = \vec{a} \\ &\quad \llcorner -\tfrac{c}{m}|\vec{v}|\,\vec{v} - g\hat{j} \end{aligned}\right] \text{4 ODEs}$$

$$\left.\begin{aligned} \Delta\vec{r} &\approx \dot{\vec{r}}\Delta t \\ \Delta\vec{v} &\approx \dot{\vec{v}}\Delta t \end{aligned}\right] \text{Euler's method}$$

$\boxed{9/6/13}$

① Continue numerical solution

② Properties of solutions of $\vec{F} = m\vec{a}$

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① Numerical example

Lecture Demo

$\boxed{9/9/13}$

Today change & Conservation of $\vec{L}, \vec{H}, E$ for a particle

start with $\vec{F} = m\vec{a}$

$\hookrightarrow \vec{F} = \vec{F}(t, \vec{r}, \vec{v})$

want to know properties of solutions

why? I. to check solutions numerically or analytically

II. Can use $\vec{L}, \vec{H}$ & E to solve simple problems

Linear Momentum

$$\vec{F} = m\vec{a}$$

$$\hookrightarrow \vec{a} = \frac{\partial \vec{v}}{\partial t}$$

$$\frac{\partial}{\partial t}\underbrace{(m\vec{v})}_{\vec{L}}$$

$$\{\vec{F} = \dot{\vec{L}}\} = \dot{\vec{L}}$$

$$\int_{t1}^{t_2} \{\} dt \Rightarrow \boxed{\int_{t_1}^{t_2} \vec{F} dt = \Delta \vec{L}}$$

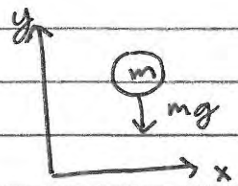
Principle of Impulse-momentum

$$\text{Impulse} = \int_{t_1}^{t_2} \vec{F} dt$$

(Linear) momentum $= m\vec{v}$

Special case: $\vec{F} = 0$

$$\int \vec{F} dt = 0 \Rightarrow \Delta \vec{L} = 0$$

$\boxed{\vec{L} = \text{const}}$ "conservation of linear momentum" ex 

<u>LMB</u>: $\vec{F} = m\vec{a}$

$$-mg\hat{j} = \dot{\vec{L}}$$

$$\{\vec{F} = m\vec{a}\} \cdot \hat{i}$$

$$\Rightarrow L_x = \text{const}$$

conservation of Lin Mom in x direction

<u>Angular Momentum</u>

$$\{\vec{F} = m\vec{a}\}$$

"scalars slide through a vector like mercury through a chicken"

$$\underset{\uparrow\, \vec{r}_{P/O}}{\vec{r}} \times \{\} \Rightarrow \vec{r} \times \vec{F} = \vec{r} \times (m\vec{a})$$

$$= m\vec{r} \times \vec{a}$$

Look at $\left[\frac{\partial}{\partial t}(\vec{r} \times \vec{v})\right]$

$$= \underbrace{\dot{\vec{r}} \times \vec{v}}_{0} + \vec{r} \times \dot{\vec{v}} = \vec{r} \times \vec{a}$$

$$\Rightarrow \vec{r} \times \vec{F} = \frac{\partial}{\partial t}\underbrace{(\vec{r} \times m\vec{v})}_{\vec{H} \neq \vec{L}}$$

$$\{\vec{r} \times \vec{F} = \dot{\vec{H}}\}$$

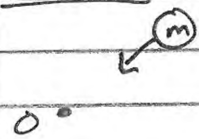
$$\int \{\}\, dt \Rightarrow \int \underset{\uparrow\ \text{moment vector}}{\vec{M}}\, dt = \Delta\vec{H}$$

Special case $\vec{M} = \vec{0}$

$$\Rightarrow \boxed{\begin{array}{l} \vec{H} = \text{const} \\ \text{conservation of angular momentum} \end{array}}$$

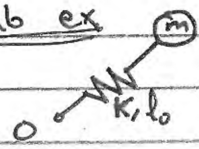
ex) <u>central force</u>: $\vec{F} = F\frac{\vec{r}}{|\vec{r}|} = F\hat{e}_r$

Sub ex



$$\vec{F} = \frac{-mMG}{r^2}\hat{e}_r = \frac{-mMG\vec{r}}{r^3}$$

Sub ex



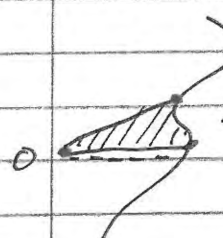
$$\vec{F} = -k(r - l_0)\hat{e}_r = \boxed{\frac{-k(r-l_0)\vec{r}}{r}}$$

Comments on & Conservation of Ang Mom

$\underbrace{\vec{M}}_{\vec{M}_{/O}} = \vec{0} \Rightarrow \underbrace{\vec{H}}_{\vec{H}_{/O}} = \text{const}$

$\Rightarrow \vec{r} \times m\vec{v} = \text{const}$

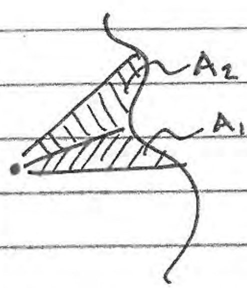
$\Rightarrow$ "Equal areas in equal times"



$\Delta A = \frac{1}{2}\underbrace{r v^{\perp}}_{|\vec{r} \times \vec{v}|} dt$

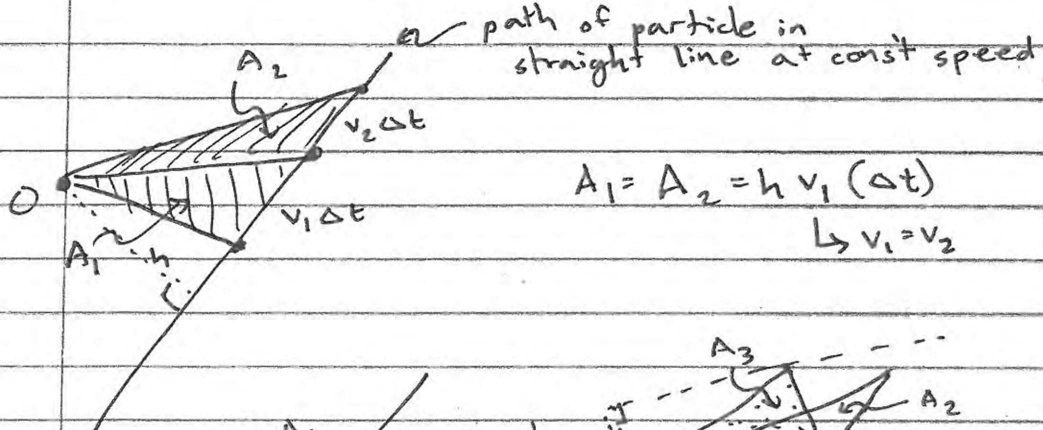
area swept per unit time is constant

$\Rightarrow \dot{A} = \text{const}$



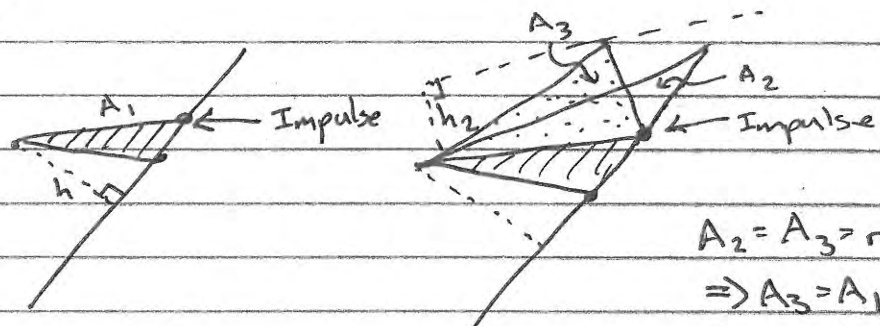
if $(\Delta t)_1 = (\Delta t)_2$ then $A_1 = A_2$

Aside: Watch Feynman Messenger Lectures "Character of Physical Law"



$$A_1 = A_2 = h\, v_1 (\Delta t)$$

$\hookrightarrow v_1 = v_2$



$A_2 = A_3 = r_2 h_2$

$\Rightarrow A_3 = A_1$

$\Rightarrow$ central force has no effect on area swept

Energy

Review this math before next lecture :

Five equivalent facts about a vector Field $\vec{F}$,

- If 1 is true, all are true
- If 1 is false, all are false

I. $\vec{F}$ is conservative

II. $\vec{F} = -\vec{\nabla} V$ $\quad V$ is a single valued potential

III. $\vec{\nabla} \vec{F} = \vec{0}$ everywhere

IV. $\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\, d\vec{r}$ = path ind

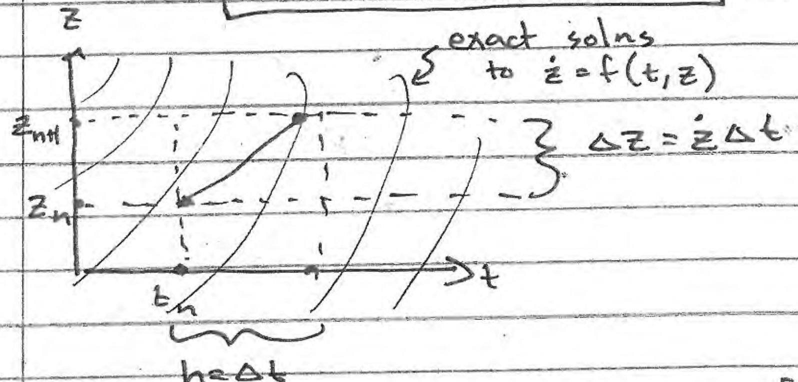
V. $\oint \vec{F} \cdot d\vec{r} = 0$ all closed loops

9/11/13

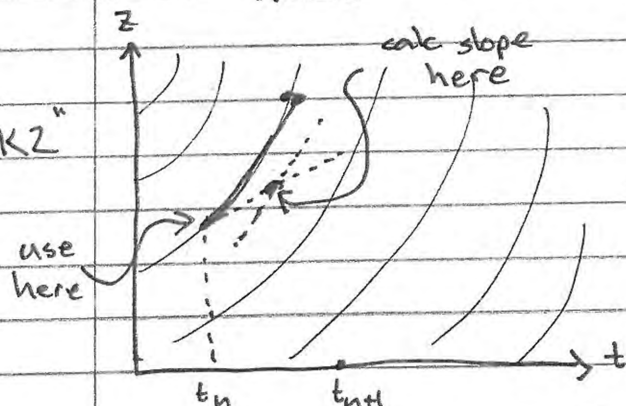
TODAY: Midpoint Method
Work Energy for a particle

RECALL: Euler method for $\dot{z} = f(t,z)$

$$\boxed{z_{n+1} = z_n + f(t_n, z_n)\cdot\Delta t}$$



"RK2"



$$z_{temp} = z_n + f(t_n, z_n)\frac{h}{2}$$

$$z_{n+1} = z_n + f\left(t_n + \frac{h}{2}, z_{temp}\right)h$$

Multiply iterations by 1000 ⇒
- Accuracy Euler's goes up by 1000
- Accuracy Midpoint goes up by 1000 000

Work + Energy

Recall

$$\int \{\vec{F} = m\vec{a}\}\,dt \Rightarrow \text{Impulse momentum}$$

$$\vec{r} \times \{\vec{F} = m\vec{a}\} \Rightarrow \text{Angular momentum}$$

Moving on:

$$\{\vec{F} = m\vec{a}\}$$

$$\{\}\cdot\vec{v} \Rightarrow \vec{F}\cdot\vec{v} = m\vec{a}\cdot\vec{v}$$

observe that $\frac{\partial}{\partial t}(v^2) = \frac{\partial}{\partial t}(\vec{v}\cdot\vec{v})$

$$= \dot{\vec{v}}\cdot\vec{v} + \vec{v}\cdot\dot{\vec{v}}$$

$$= 2\dot{\vec{v}}\cdot\vec{v}$$

$$= 2\vec{a}\cdot\vec{v}$$

$$\Rightarrow \underbrace{\vec{F}\cdot\vec{v}}_{P=\text{power}} = \frac{\partial}{\partial t}\underbrace{\left(\tfrac{1}{2}mv^2\right)}_{E_k=\text{Kinetic Energy}}$$

$$\boxed{P = \dot{E}_k}$$

$$\int\{\}\,dt \Rightarrow \int_{t_1}^{t_2} P\,dt = \Delta E_k$$

$$E_{k2} - E_{k1}$$

$$\int_{t1}^{t2} P\,dt = \int_{t1}^{t2} \vec{F}\cdot\underbrace{\vec{v}\,dt}_{d\vec{r}}$$

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \sum \vec{F}\cdot\Delta\vec{r} = \text{"force}\cdot\text{distance"}$$

W ← work

"$F\cdot\Delta x$"

$$\boxed{W = \Delta E_k}$$

Special cases: Conservative Forces

$\vec{F}(t,\vec{r},\vec{v}) = \vec{F}(\vec{r})$ and they are conservative

Conservative $\Leftrightarrow$ some scalar $V(x,y) = V(\vec{r})$ exist with

$$\vec{F} = -\vec{\nabla} V \Leftrightarrow F_x = \frac{-\partial E_p}{\partial x},\ F_y = \frac{-\partial V}{\partial y}$$

$V = E_p$ ← potential energy

ex) near earth gravity: $\vec{F} = -mg\hat{y}$

$$F_x = 0$$

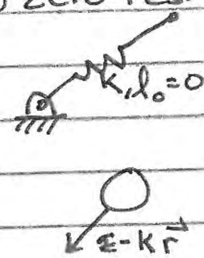
$$F_y = -mg$$

$E_p = mgy$

check: $\frac{-\partial E_p}{\partial x} = \frac{-\partial mgy}{\partial x} = 0$ ✓

$$\frac{-\partial E_p}{\partial y} = -mg \quad ✓$$

ex) zero-rest-length spring

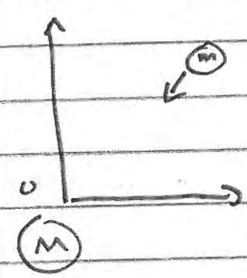


$$\vec{F} = -k\vec{r}$$

$$E_p = \frac{1}{2} k r^2$$

check: $-\vec{\nabla} E_p = -k\overbrace{r\,\hat{e}_r}^{\vec{r}}$

ex) Inverse square gravity



$\vec{F} = \frac{-mMG}{r^2}\hat{e}_r \qquad E_p = \frac{-mMG}{r}$

check: $-\vec{\nabla} E_p = -\left(\frac{-mMG}{r}\right)\hat{e}_r$

$$-\vec{\nabla} E_p = \frac{-mMG}{r^2}\hat{e}_r \quad ✓$$

ex) $\vec{F} = y\hat{i} - x\hat{j}$

$F_x = y$

$F_y = -x$

$F_x = \frac{-\partial E_p}{\partial y} \qquad F_y = \frac{\partial E_p}{\partial y}$

$\frac{-\partial F_x}{\partial y} = -1 \qquad \frac{\partial F_y}{\partial x} = 1$

$$\frac{\partial^2 E_p}{\partial y \partial x} \neq \frac{\partial^2 E_p}{\partial x \partial y}$$

$\Rightarrow$ No such E_p exists

$\vec{F}$ is not conservative

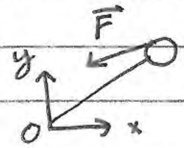
9/13/13

TODAY

1) W-E (con't)

2) Multiplicative systems

Consider $\vec{F}(t, \vec{r}, \vec{v}) = \vec{F}(\vec{r})$

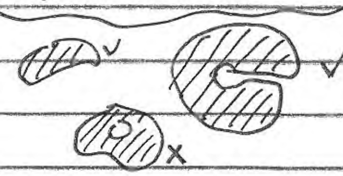


ex) $\vec{F} = -k r \hat{e}_r$
$= -k(x\hat{i} + y\hat{j})$

ex) $\vec{F} = k r \hat{e}_\theta$
$= k(-y\hat{i} + x\hat{j})$

$\vec{F}$ is conservative

In a simply-connected region of interest



$\Updownarrow$

I. $E_p(\vec{r})$ exists with

$\vec{F} = -\vec{\nabla} E_p$

$F_x = \frac{-\partial E_p}{\partial x}$, $F_y = \frac{-\partial E_p}{\partial y}$

ex 1) $E_p = kr^2/2$, ex2) $E_p =$ no such thing

$\Updownarrow$

II. $\vec{\nabla} \times \vec{F} = \vec{0}$ ex1) $0 = 0$ ✓

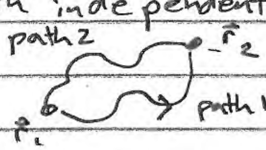
$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$ ex2) $-1 \neq 1$ X

$\Updownarrow$

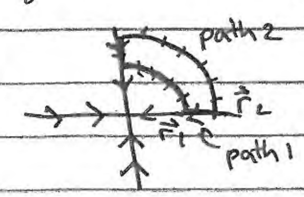
III. For any pair $\vec{r}_1$ & $\vec{r}_2$

and any pair of paths between them

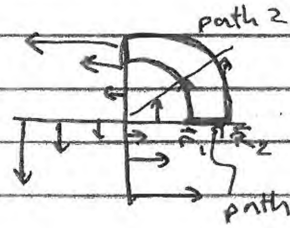
$$\underbrace{\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}}_{\text{the work integral}}$$ is path independent



ex1)



ex 2)



$\int \vec{F} \cdot d\vec{r} = 0$

path 1: no work

path 2: lots of work

$\int \vec{F} \cdot d\vec{r} \neq 0$

$$\Updownarrow$$

IV. $\oint \vec{F}\cdot d\vec{r} = 0$ all closed paths

"net work of force when going on a "round trip" = 0"

ex1) $\oint \vec{F}\cdot d\vec{r} = 0$ ex2) $\oint \vec{F}\cdot d\vec{r} \neq 0$

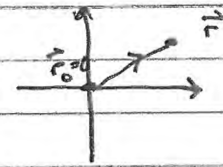
$$\Updownarrow$$

V. $E_p(\vec{r}) = \int_{\vec{r}_0}^{\vec{r}} \vec{F}\cdot d\vec{r}$ $\vec{r}$ of your choice

path of your choice

ex1) $\vec{F} = k\vec{r}$

$$\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'$$



parameterize: $\vec{r}\,' = t\vec{r}$

$0 \le t \le 1$

$$d\vec{r}\,' = d(t\vec{r}) = \vec{r}\,dt$$

$$\vec{F}(\vec{r}\,') = k\vec{r}\,' = kt\vec{r}$$

$$\vec{F}(\vec{r}\,')\cdot d\vec{r}\,' = (kt\vec{r})\cdot(\vec{r}\,dt)$$

$$= r^2 k\, t\,dt$$

$$-\int_{\vec{r}_0}^{\vec{r}} \vec{F}\cdot d\vec{r}\,' = -\int_0^1 k r^2 t\,dt = -kr^2\int_0^1 t\,dt$$

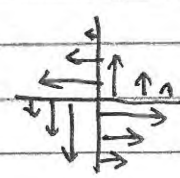
$$= -kr^2/2 = E_p$$

$\vec{F} = -y\hat{i} + x\hat{j}$ Non-conservative

valuable

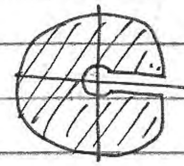
Subtle ex)

$$\vec{F} = \frac{1}{r}\hat{e}_\theta$$

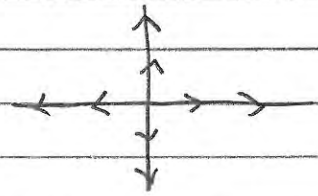


$\vec{\nabla}\times\vec{F} = \vec{0}$

everywhere but $\vec{0}$



conservative force field if does NOT include origin ex) $\vec{F} = k\vec{r}$



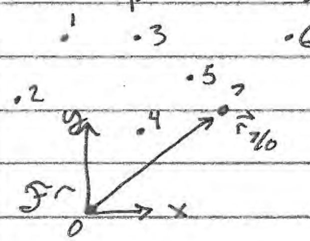
Conservative

- closed paths are zero
- infinite energy over infinite length

Multiparticle

$$\vec{F}^i = \vec{F}^i(t, \vec{r}^1, \vec{r}^2, \dots \vec{r}^i, \vec{v}^1, v^2, \dots \vec{v}^n)$$

Multi-particle systems $\boxed{9/16/13}$



$\vec{r}_{75} = \vec{r}_{5/7} = \vec{r}_5 - \vec{r}_7$

$\vec{F}_{7} = -m_7\vec{a}_7$

need to know $\vec{r}_i$ and $\vec{v}_i$ of other particles

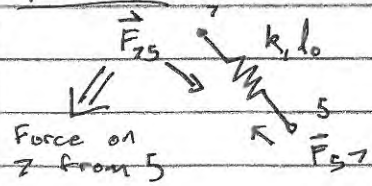
$$L = \vec{F}_7(t, \vec{r}_1, \vec{r}_2 \dots \vec{r}_7, \vec{v}_1, \vec{v}_2, \dots \vec{v}_7)$$

$$\text{eg: } \vec{F}_7 = -m_7 g\hat{y} + \underbrace{k_{75}\left(|\vec{r}_5 - \vec{r}_7| - l_0\right)\frac{\vec{r}_5 - \vec{r}_7}{|r_5 - r_7|}}_{\text{spring between particle 5 and 7}}$$

if n particles need 4n 1st order ODEs

given initial state $\Rightarrow$ predict future, (Laplace, quantum mech, chaos)

For HW ..



$\vec{F}_{75} = T_{75}\hat{e}_{75}$

$T_{75} \to k(l_{75} - l_0)$

$\hat{e}_{75} \to \dfrac{\vec{r}_5 - \vec{r}_7}{|\vec{r}_5 - \vec{r}_7|}$

$\hookrightarrow l_{75} = l_{57} = |\vec{r}_{5/7}| = |\vec{r}_7 - \vec{r}_5|$

Linear Momentum

one particle: $\vec{F}_i = m_i \vec{a}_i$

$\hookrightarrow F_i^{tot}$ $\hookrightarrow a_{i/\mathcal{F}}$

add up particles $\sum \vec{F}_i = \sum m_i \vec{a}_i$

$$\sum \left(\vec{F}_i^{int} + \vec{F}_i^{ext} \right) = \frac{\partial}{\partial t} \underbrace{\sum m_i \vec{v}_i}_{\vec{L}}$$

$$\underbrace{\sum \vec{F}_i^{int}}_{= 0 \text{ for any of three choices}} + \sum \vec{F}_i^{ext} = \frac{\partial}{\partial t} \vec{L}$$

① Classical reason (see book) assumes central pairwise forces:

a) $F_i^{int} = \sum F_{ij}^{int}$

b) $\vec{F}_{ij} = -\vec{F}_{ij}$

c) $\vec{F}_{ij} = \vec{F}_{ij} \frac{(\vec{r}_j - \vec{r}_i)}{|\;|}$

accurate in macroscopic scale, but forces don't act like this on micro scale

② $\vec{F}^{int} = 0$ by assumption — both imply each other

③ $\underbrace{\vec{F}_i^{ext} = \dot{\vec{L}}}_{\text{can be shown experimentally}}$ by assumption

Look at $\vec{L}$

$$\vec{L} = \sum m_i \vec{v}_i = \cancel{\sum m_i \vec{r}_i}$$

$$= \frac{\partial}{\partial t} \sum m_i \vec{r}_i$$

$$= \frac{\partial}{\partial t} (m_{tot} \vec{r}_G)$$

center of mass

$$\vec{r}_G = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

$\Downarrow$

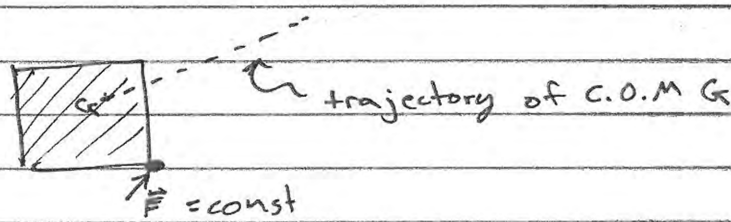
$m_{tot} \vec{r}_G \sum m_i \vec{r}_i$

LMB: $\boxed{\vec{F}^{ext} = m \ddot{\vec{r}}_G}$

$\vec{F} = m\vec{a}$ is correct & precise

$\vec{F} = \sum \vec{F}_i^{ext}$ $\hookrightarrow \vec{a} = \vec{a}_{G/\mathcal{F}}$

Ex) Start from rest



Angular Momentum

one particle $\{\vec{F}_i = m_i \vec{a}_i\}$

$$\sum \vec{r}_{i/c} \times \{\,\} \Rightarrow \underbrace{\sum \vec{r}_{i/c} \times \left(\vec{F}_i^{\,int} + \vec{F}_i^{\,ext}\right) = \sum \vec{r}_{i/c} \times m_i \vec{a}_{i/\mathcal{F}}}_{\text{ALWAYS TRUE!}}$$

Assume $\sum \vec{r}_{i/c} \times \vec{F}_i^{\,int} = 0$

$\Rightarrow$ internal forces have no net torque for any C

$$\sum \vec{r}_{i/c} \times \vec{F}^{ext} = \sum \vec{r}_{i/c} \times (m_i \vec{a}_{i/\mathcal{F}})$$

9/18/13

TODAY

* Multi-particles (cont'd)
* C.O.M. = G

Recall: 3 ways to go from 1 particle to many

i) Assume LMB & AMB apply to any "closed" system

$\sum \vec{F}^{ext} = \dot{\vec{L}}$

↳ $\vec{L} = \sum m_i \vec{v}_i$

↳ $\equiv \vec{v}_{i/\mathcal{F}}$

$\sum \vec{M}_{/c}^{ext} = \sum m_i \vec{r}_{i/c} \times \vec{a}_{i/\mathcal{F}}$

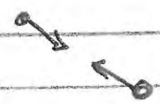
2) (Equiv to 1)

Assume

$$\sum \vec{F}^{\,int} = \vec{0} \quad \& \quad \sum \vec{M}_{/c}^{\,int} = \vec{0}$$

$$\& \quad \vec{F}^{\,TOT} = m\vec{a} \quad \text{for each particle}$$

3) Classical approach:



pairwise central equal & opposite forces

$$\& \quad \vec{F} = m\vec{a}$$

Problematic because

a) microscopic assumption
(none of our business)

b) wrong physics

c) Bad macroscopic predictions, e.g. $\nu = 1/4$
↖ poissons ratio

Modern physics takes ① as a postulate
with $F = ma$ as a special condition

Ang Mom

$$\boxed{\sum \vec{M}_{/c} = \sum \vec{r}_{i/c} \times (m\vec{a}_{i/F})} \quad \text{AMB}$$

$$\sum \vec{M}_{/c} = \frac{d}{dt}\left(\vec{H}_{/c}\right)$$

↑ if we define $\vec{H}_{/c}$ appropriately

What are good defs of $\vec{H}_{/c}$?

① C is a $\underbrace{\text{fixed pt}}$ → fixed in any Newtonian frame

$$\vec{v} = \vec{v}_{/c} = \vec{v}_{/F}$$

$$\vec{H}_{/c} = \sum \vec{r}_{i/c} \times m\vec{v}_{i/c}$$

→ $\vec{v}_{i/F}$

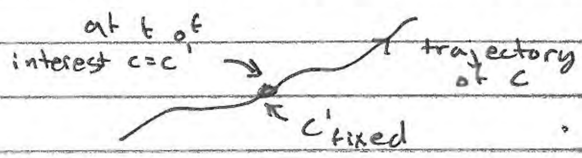
② $c = G = COM = \sum \vec{r}_i m_i / M_{tot}$

$$\vec{H}_{/G} = \sum \vec{r}_{i/G} \times m_i \vec{v}_{i/G}$$

$$\Rightarrow \sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$$

③ $\vec{H}_{/c} = \vec{r}_{i/c'} \times m_i \vec{v}_{i/F}$

c' is a fixed pt inst. coinciding with C



Summary

define $\vec{H}_{/c}$

If $\frac{\partial}{\partial t}\vec{H}_{/c} = \sum \vec{r}_{i/c} \times m\vec{a}_{i/F}$ good

else bad

Energy

$$\vec{F}_i = m_i \vec{a}_i$$

$$\vec{v}_i \cdot \vec{F}_i = m_i \vec{v}_i \cdot \vec{a}_i$$

Power of all forces equal rate of change of kinetic energy

$$\sum \vec{v}_i \cdot \vec{F}_i = \sum m_i \vec{v}_i \cdot \vec{a}_i$$

$\vec{F}_i = \vec{F}_i^{int} + \vec{F}^{ext}$

$$\boxed{P^{int} + P^{ext} = \dot{E}_k}$$

$$E_k = \sum \tfrac{1}{2} m_i v_i^2$$

Take cons. forces (int + ext)
+ associate pot. Energy with them

$$P_{nc}^{ext} + P_{nc}^{int} = \dot{E}_k + \dot{E}_p$$

↳ non-conservative

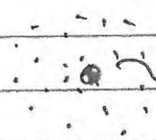
$\dot{E}_p$ ↳ $\sum E_p^{int} + E_p^{ext}$

Often assumed $-P_{nc}^{int} = \dot{D} \geq 0$
↳ dissapation

$$P_{nc}^{ext} - \dot{D} = \dot{E}_{tot}$$

↳ rocket engines, or things pushing on a system

Center of Mass



G, COM = average position of mass of system

$$m_{tot}\vec{r}_G = \sum \vec{r}_i m_i$$

Use COM to simplify $\dot{\vec{L}}$, $\vec{H}_{/C}$, E_k

$$\dot{\vec{L}} = \sum m_i \vec{a}_i$$
$$= \frac{d}{dt}\left(\sum m_i \vec{v}_i\right) = \frac{d}{dt}\left(m_{tot}\vec{v}_G\right)$$

$$\boxed{\vec{L} = \frac{d}{dt}\left(\vec{r}_G M_{TOT}\right)}$$

$\vec{H}_{/C} = \sum \vec{r}_{i/C} \times m_i \vec{v}_{i/C}$ (C is a fixed point)

*

Aside:

$$\vec{L} = \sum m_i \vec{v}_i$$

$\vec{v}_i$ ↳ $\vec{v}_G + \vec{v}_{i/G}$

$$= \sum m_i v_G + \sum m_i \vec{v}_{i/G}$$

$$\vec{L} = \sum m_i v_G + \sum m_i \vec{v}_i - \sum m_i \vec{v}_G$$
$$= \sum m_i \vec{v}_G + m_{tot}\vec{v}_G - m_{tot}\vec{v}_G$$
$$= \sum m_i \vec{v}_G = m_{tot}\vec{v}_G$$

Back to *

$$\vec{r}_{i/c} = \vec{r}_{G/c} + \vec{r}_{i/G}$$

$$\vec{v}_{i/c} = \vec{v}_{G/c} + \vec{v}_{i/G}$$

$$\vec{H}_{/c} = \sum \left(\vec{r}_{G/c} + \vec{r}_{i/c}\right) \times \left(\vec{v}_G + \vec{v}_{i/G}\right) m_i$$

2 of 4 terms are $\varnothing$ (see $\vec{L}$ calc)

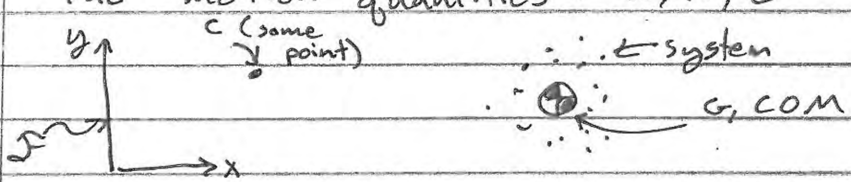
$$\Rightarrow \quad \vec{H}_{/c} = \vec{r}_{G/c} \times m_{TOT} \vec{v}_G + \sum \vec{r}_{i/G} \times \vec{v}_{i/G}\, m$$

$$\boxed{\vec{H}_{/c} = \vec{H}_{G/c} + \vec{H}_{/G}}$$

TODAY — 9/20/13

Multi-particles (con't):

The motion quantities: $\vec{L}, \vec{H}, E$



Linear Momentum

$$\vec{L} = \sum m_i \vec{v}_i \underset{\uparrow}{=} \frac{d}{dt}\left(m_{TOT} \vec{r}_G\right)$$

simplification

Any system $\underline{\underline{\text{is}}}$ a particle

$$\boxed{\vec{F}^{TOT} = m\vec{a}_G}$$

Angular Momentum

$$\vec{H}_{/c} = \sum \vec{r}_{i/c'} \times m \vec{v}_{i/\mathcal{F}}$$

where c' is fixed in $\mathcal{F}$
and coinciding with C

$$\vec{M}_{/c} = \underbrace{\sum \vec{r}_{i/c} \times m_i \vec{a}_{i/\mathcal{F}}}_{\dot{\vec{H}}_{/c}}$$

useful fact:

$$\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$$

$$\left.\vec{H}_{/G} = \sum \vec{r}_{i/G} \times m_i \vec{v}_{i/G}\right\} \text{favorite in physics classes}$$

* $\underline{H}_{/c} = \frac{\partial}{\partial t}\left[\text{There is no such thing!}\right]$

interesting things

*

Aside: Falling cat problem
Drop cat upside down. Angular momentum stays zero, but cat rotates

$$\vec{H}_{/c} = \underbrace{\vec{r}_{G/c} \times m_{tot} \vec{V}_{G/c}}_{\vec{H}_{G/c}} + \underbrace{\sum \vec{r}_{i/G} \times m_i \vec{v}_{i/G}}_{\vec{H}_{/G}}$$

$\vec{H}_{G/c}$	$\vec{H}_{/G}$
rotation about point C	rotation about its own axis
ex) Earth's orbit	ex) Earth's rotation

$$\vec{M}_{/C} = \dot{\vec{H}}_{G/C} + \dot{\vec{H}}_{/G}$$

$$\vec{r}_{G/C} \times \vec{F}^{TOT} + \vec{M}_{/G} = \text{''} + \text{''}$$

$$\Rightarrow \boxed{\begin{array}{c} \vec{r}_{G/C} \times \vec{F}^{TOT} = \dot{\vec{H}}_{G/C} \\ \vec{M}_{/G} = \dot{\vec{H}}_{/G} \end{array}}$$

<u>Energy E_k</u>

Konig's Thm.

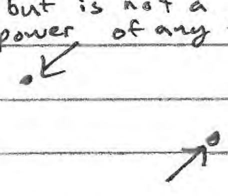
$$E_k = \frac{1}{2} m_{tot} \underset{\dot{r}_G}{v_G}^2 + \underbrace{\frac{1}{2} \sum m_i \left[(\vec{v}_i - \vec{v}_G)\right]^2}_{\frac{1}{2} \sum m_i \vec{v}_{i/G} \cdot \vec{v}_{i/G}}$$

★ can't break into 2 simple eqs, <u>BUT</u> we do have 1:

$$\underbrace{\vec{F}^{TOT} \cdot \vec{v}_G}_{\text{"power"}} = \frac{\partial}{\partial t} \underbrace{E_{GK}}_{\frac{1}{2} m v_G^2}$$

has dimensions of power but is not a physical power of any forces

ex)



- total force = 0
- rate of change of kinetic energy = 0

9/23/13

TODAY

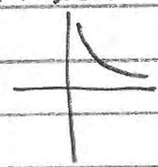
Vibrations (1 of ≈12)

sinusoidally-forced damped harmonic oscillator

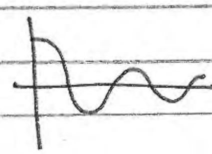
$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega_f t)$$

has

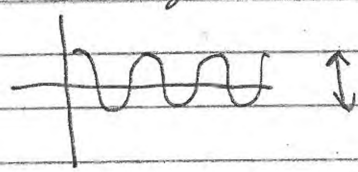
① Transient soln



or



② And steady state soln.



resonance & all that

Core concept:

harmonic oscillator: $m\ddot{x} + kx = 0$

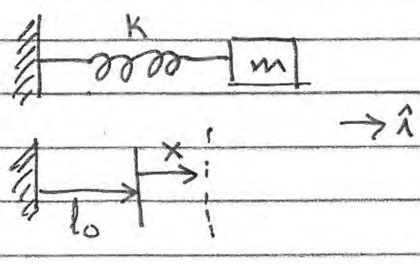
Three different frequencies: ω_f that provides max peaks

ω_n natural frequency

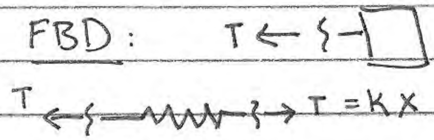
frequency when unforced

All three are close

Harmonic Oscillator (spring-mass system)



FBD:



LMB: $\vec{F} = m\vec{a}$

$$\{ -T\hat{\imath} = m\ddot{x}\hat{\imath} \}$$

$$\{ \} \cdot \hat{\imath} = -T = m\ddot{x}$$

$$m\ddot{x} = -kx$$

$$\boxed{\ddot{x} = \frac{-k}{m}x} \leftarrow \text{form for numerical soln}$$

$$\boxed{m\ddot{x} + kx = 0} \leftarrow \text{form for analytical soln}$$

Step 1 "know" the solution

$$\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= \frac{-k}{m}x \end{aligned}} \leftarrow \text{set up for computer (state space)}$$

Soln $x = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

$\omega_0 = \sqrt{k/m}$ (natural frequency)

Another approach

Guess: $x = e^{\lambda t} \Rightarrow m\lambda^2 + \cancel{c\lambda}^{0} + k = 0$

$\lambda = \ldots \pm i\sqrt{k/m} \Rightarrow x = e^{i\omega_0 t}$

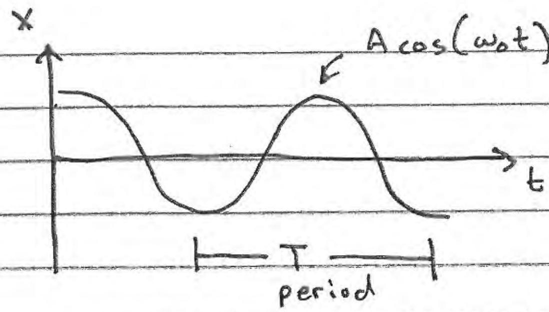
$x = A\,\mathrm{Re}(e^{i\omega t}) + B\,\mathrm{Im}(e^{i\omega t})$

or

$x = Ce^{i\omega t} + De^{-i\omega t}$

Alternate: multiply both sides by $\dot{x}$, cons of energy, integral .... sin/cos

$A\cos(\omega_0 t) + B\sin(\omega_0 t)$



$$\boxed{\omega_0 T = 2\pi}$$

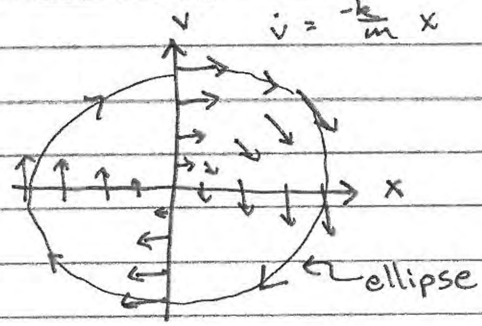
$T = \frac{2\pi}{\omega_0} \Longleftrightarrow \omega_0 = \frac{2\pi}{T}$

ω_0 : angular frequency (rad/s)

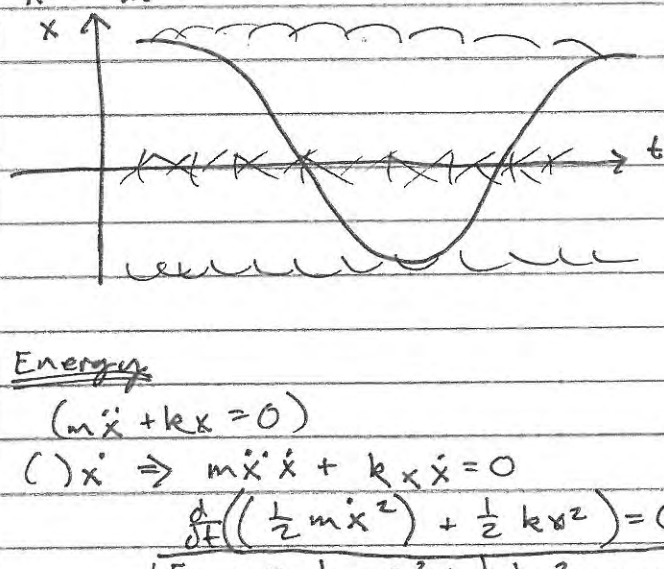
$f = \frac{1}{T} = \frac{\omega_0}{2\pi}$ = frequency (cycles/s)

Phase Plane $\quad \dot{x} = v$

$\dot{v} = \frac{-k}{m} x$



$\ddot{x} = \frac{-k}{m} x$



Energy

$(m\ddot{x} + kx = 0)$

$(\,)\dot{x} \Rightarrow m\ddot{x}\dot{x} + kx\dot{x} = 0$

$\frac{d}{dt}\left(\left(\frac{1}{2}m\dot{x}^2\right) + \frac{1}{2}kx^2\right) = 0$ derivation of

$$\boxed{E_{tot} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \text{const}}$$ conservation of energy

$\Rightarrow$ amplitude of oscillation is CONST

Start w/ cons of energy

Derive ODE

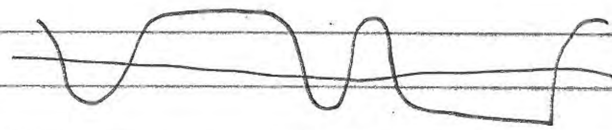
$$E_{TOT} = E_k + E_p$$

$$E_{TOT} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

$$\frac{d}{dt}(E_{tot}) = 0 \Rightarrow m\dot{x}\ddot{x} + kx\dot{x} = 0$$

$$\text{if } \dot{x} \neq 0 \Rightarrow \boxed{m\ddot{x} + kx = 0}$$

[satisfies cons of energy if you grab mass at extremes



Using only cons of energy allows "unatural" solutions

Lagrange Eqs

1 DOF: q

L.E.:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\mathcal{L}}{\partial \dot{q}} = 0$$

$$\mathcal{L} = \mathcal{L}(q, \dot{q}) = E_k - E_p$$

$E_p = \frac{1}{2} k x^2 \qquad x = q$

$E_k = \frac{1}{2} m \dot{x}^2$

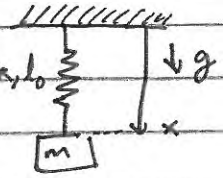
$$\mathcal{L} = \underbrace{E_k - E_p}_{\substack{\text{most books} \\ \text{"T - V"}}} = \underbrace{\frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2}_{\mathcal{L}(x, \dot{x})}$$

$$\frac{\partial \mathcal{L}}{x} - \frac{d}{\partial t}\frac{\partial \mathcal{L}}{\dot{x}}$$

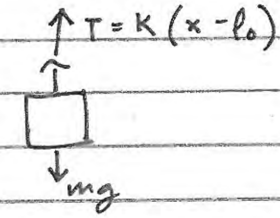
$$-kx - \frac{d}{\partial t}(+m\dot{x}) = 0 \Rightarrow \boxed{kx + m\ddot{x} = 0}$$

9/25/13 | Harmonic Oscillator (con't)

ex)



FBD



LMB $\quad m\ddot{x} = mg \cancel{-k(x-l_0)} - k(x - l_0)$

$$m\ddot{x} + kx = mg + kl_0$$

$$\ddot{x} + \frac{k}{m}x = g + \frac{k}{m}l_0$$

Method 1

$$x_{general} = x_h + x_p$$

x_p ← any "particular" solution

x_h ← general homogeneous solution

$$\ddot{x}_h + \frac{k}{m}x_h = 0 \Rightarrow x_h = A\cos\left(\sqrt{k/m}\,t\right) + B\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

In general, particular solution resembles forcing function

$$x_p = \frac{mg}{k} + l_0$$

$$x = \frac{mg}{k} + l_0 + A\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + B\sin\sqrt{\tfrac{k}{m}}\,t$$

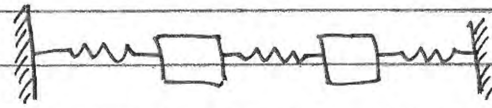
Method 2 $\quad$ Define: $z = x - \left(\frac{mg}{k} + l_0\right)$

$$\Rightarrow \ddot{z} + \frac{k}{m}z = 0$$

soln. $z = A\cos(\;) + B\sin(\;)$

Summary

In the subject of vibrations, pos is generally measured relative to equilibrium

ex) 

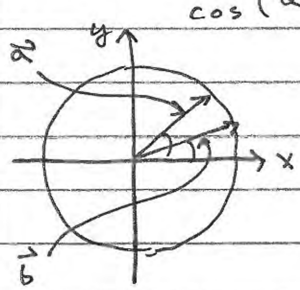
takes effort to find equilibrium
$\Rightarrow$ most vibrations textbooks setup problems already in equilibrium

TWO forms of soln.

$$x = A\cos\sqrt{k/m}\,t + B\sin\sqrt{k/m}\,t$$
$$= C\cos\left(\sqrt{k/m}\,t - \delta\right)$$

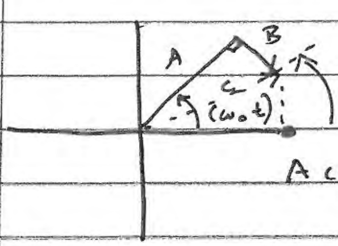
why?

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$



$$\left.\begin{aligned}\vec{a}\cdot\vec{b} &= |\vec{a}|\,|\vec{b}|\cos(\alpha-\beta)\\ &= a_x b_x + b_y a_y\end{aligned}\right] \Rightarrow$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$



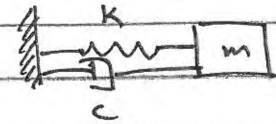
$C = \sqrt{A^2+B^2}$

$\longrightarrow\ = \omega_0 t - \operatorname{atan}(B/A)$

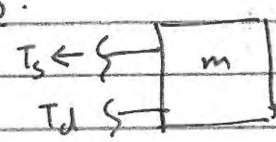
$A\cos(\omega_0 t) + B\sin(\omega_0 t)$

$$C = \sqrt{A^2+B^2}\,,\quad \delta = \begin{cases}\operatorname{atan}(B/A)\\ \operatorname{atan2}(A,B)\end{cases}$$

Damped Harmonic Oscillator



FBD:



$$T_s = kx, \quad T_d = c\dot{x}$$

LMB: $\sum F_x = m\ddot{x}$

$$-c\dot{x} - kx = m\ddot{x}$$

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0}$$

Solve by guessing

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

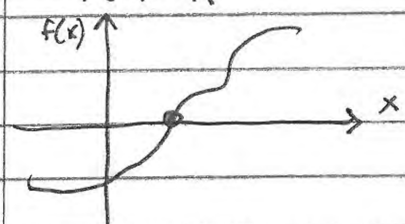
$\boxed{9/27/13}$ Aside on root finding and periodic orbits

$$x \longrightarrow \boxed{\text{black box}} \longrightarrow f(x)$$

puzzle: $\underbrace{\text{find } x \text{ so that } f(x) = 0}_{\text{root finding}}$

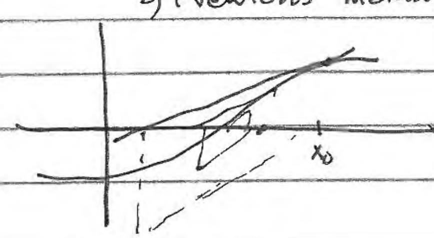
ex) $x \in \mathbb{R}^1$

$f(x) \in \mathbb{R}^1$



2 approaches:

1) bisection
2) Newton's method



Root finding w/ multi-variables

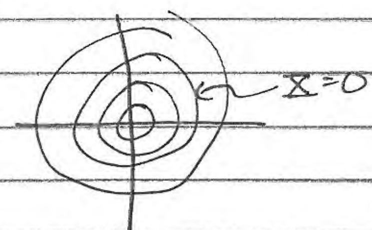
1) bisection doesn't work
2) Newtons method is O.K.

ex) "X" $= \mathbb{R}^2$

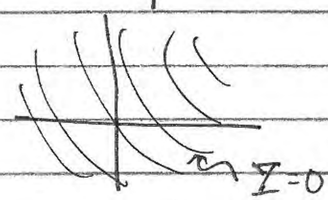
$f = \mathbb{R}^2$

$$\left.\begin{aligned} X &= X(x,y) \\ Y &= Y(x,y) \end{aligned}\right\} = f$$

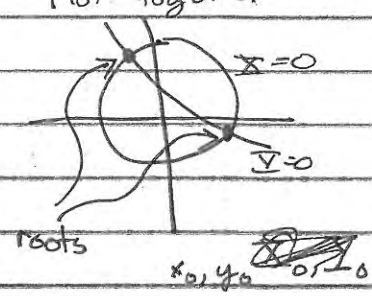
X level lines



Y ~~lote~~ level lines



Plot together



Initial Guess:

x_0, y_0

$\Rightarrow X_0, Y_0$

$\Rightarrow$ approximation

$$X = X_0 + (x - x_0)\frac{\partial X}{\partial x} + (y - y_0)\frac{\partial X}{\partial y}$$

$$Y = Y_0 + (x - x_0)\frac{\partial Y}{\partial x} + (y - y_0)\frac{\partial Y}{\partial y}$$

To find x, y solve for $x_1 y_1$

$$\left.\begin{aligned} 0 &= X_0 + \frac{\partial X}{\partial x}(x_1 - x_0) + \frac{\partial X}{\partial y}(y_1 - y_0) \\ 0 &= Y_0 + \frac{\partial Y}{\partial x}(x_1 - x_0) + \frac{\partial Y}{\partial y}(y_1 - y_0) \end{aligned}\right\} \text{derivatives evaluated at } x_0, y_0$$

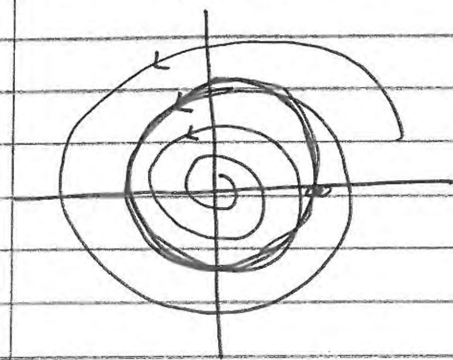
$$\overbrace{\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} + -\underbrace{\begin{bmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{bmatrix}}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}^{\text{known} = B} = -\begin{bmatrix} J \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Solve $Ax = B$ for x

$\Rightarrow x$

repeat etc $\Rightarrow$ Newton's method

Find periodic solns of ODEs



fix one variable (e.g. $y=0$)
"picking a Poincare section"

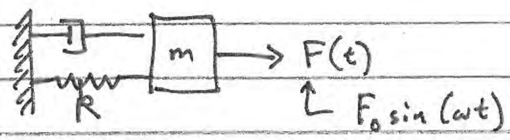
guess: x_0, t_f
eqns $\Rightarrow x_1, y_1$

evaluate $f = \begin{bmatrix} x_1 - x_0 \\ y_1 - 0 \end{bmatrix}$

$y_1 - 0$

9/30/13 Vibes (cont'd)

1 DoF damped - forced vibrations



$$\boxed{m\ddot{x} + c\dot{x} + kx = F(t)} \quad \star$$

Various ways to shake something

- applying force
- shaking base

$\Rightarrow F(t)$

$$\star \Rightarrow \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin(\omega t)$$

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \bar{F}_0 \sin(\omega t)} \quad \star$$

$\omega_n = \sqrt{k/m}$ $\quad \zeta$ = damping ratio $= \frac{c}{2\sqrt{km}}$

$$2\zeta\omega_n = \frac{c}{m}$$

Soln of $\star$ $x(t)$

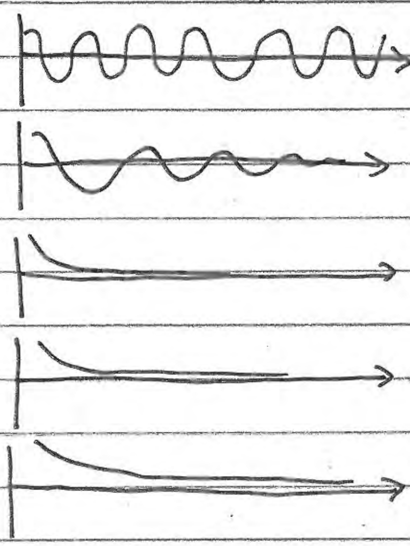
$$x(t) = x_h + x_p$$

x_p ← particular soln (steady state response)

x_h ← homogeneous soln (transient response)

Homog. (transient) soln

set $F(t) = 0$



$c = 0$, $\zeta = 0$ — no damping; c small } underdamped

c = critical } get to equilibrium quickly $\Rightarrow$ roughly fastest decay

c big ish; c big, ζ big } overdamped

$\zeta = \frac{c}{c_{crit}}$ critical: $\sqrt{c^2 - 4km} = 0$

$c_{crit} = 2\sqrt{km}$

x_h = Lin comb of solns.

each is $\Re(e^{\lambda t})$

λ roots of $\lambda = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$

$$\lambda = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$$

$$\boxed{\lambda = \omega_n\left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]}$$

2 cases: real roots, complex roots

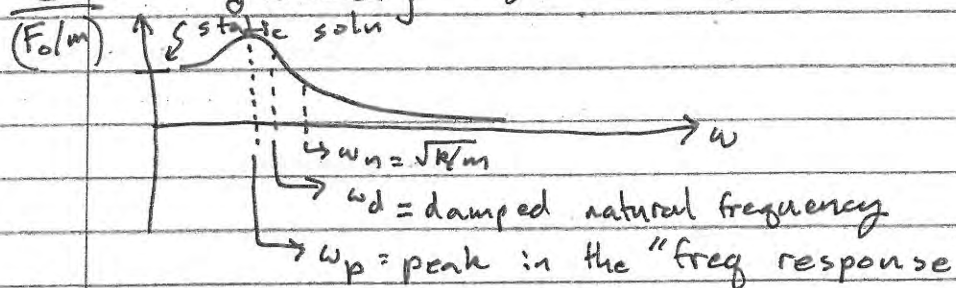
steady state soln: x_p

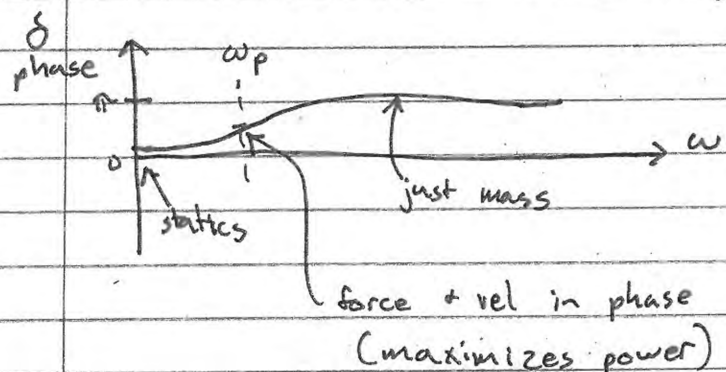
$F(t)$: very slow, amplitude $\propto k$, movement in phase with force

$F(t)$: very fast, movement completely opposite phase to force steady state soln: x_p

$$x_p = A\cos(\omega t) + B\sin(\omega t)$$
$$= C\sin(\omega t - \delta) \quad C = \sqrt{A^2+B^2}$$

$$\left.\begin{array}{l} A = \sim\sim \\ B = \sim\sim \\ C = \sim\sim \\ \delta = \sim\sim \end{array}\right\} \left.\begin{array}{ll} \text{Tongue} & 2.6.14 \\ \text{RP} & 10.33 \\ \text{Taylor} & 5.68 \end{array}\right\} \text{lots of algebra}$$



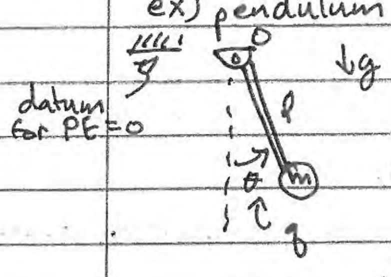


Picking up on Lagrange eqns

For some class of probs, L.E. replace Newton's Laws

1 DoF systems (see prev. lecture)

ex) pendulum



$E_p = -mgh = mg\,l\cos\theta = \text{"}V\text{"}$

$E_k = \frac{1}{2}m(l\dot{\theta})^2 = \text{"}T\text{"}$

L.E.:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0$$

$$\mathcal{L} = E_k - E_p = 0$$

L.E.: $\frac{\partial \mathcal{L}}{\partial q} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0$

$$\mathcal{L} = E_k - E_p = \frac{1}{2} m^2 l^2 \dot{\theta}^2 + mgl\cos\theta$$

$$\Rightarrow \left(-mgl\sin\theta\right) - \frac{\partial}{\partial t}\left(m l^2 \dot{\theta}\right) = 0$$

$$-\cancel{m}gl\sin\theta - \cancel{m}l^2\ddot{\theta} = 0$$

$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$ The pendulum eqn

has soln $\boxed{\theta = 0}$

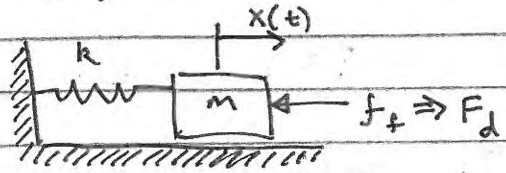
Near equilibrium

$\theta << 1$

$\Rightarrow \boxed{\ddot{\theta} + \frac{g}{l}\theta = 0}$ linearized pend eqn

10/2/13 Guest Lecture from Dr. Garcia

Damping Models Tongue 2.10 - 2.12



$$m\ddot{x} + kx = F_d$$

what generates damping?

$$F_d = -c\dot{x}$$

What else?

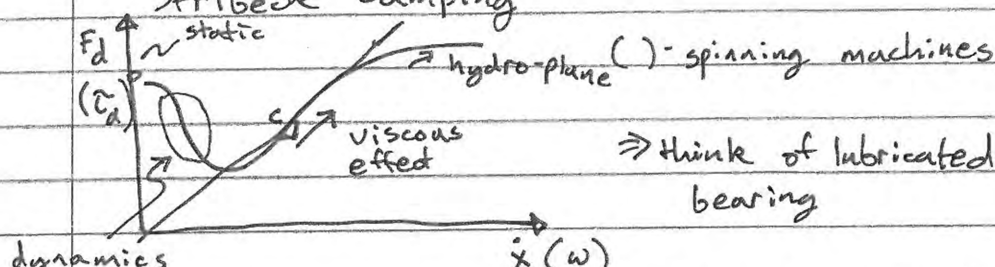
$F_d = -\mu_s N$, $\hookrightarrow N = mg$ } static friction

$F_d = -\mu_d N$ } dynamic friction

$F_d = \frac{1}{2}\rho\dot{x}^2 A_{block} c_d$ } air drag

$F_d = -k\beta i$ } hyteretic damping in the spring

Stribeck Damping



⇒ think of lubricated bearing

Sec 2.10 (Tongue)

Identification of damping + natural frequency

- if damping is small, $\Rightarrow \omega_d \approx \omega_n$

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

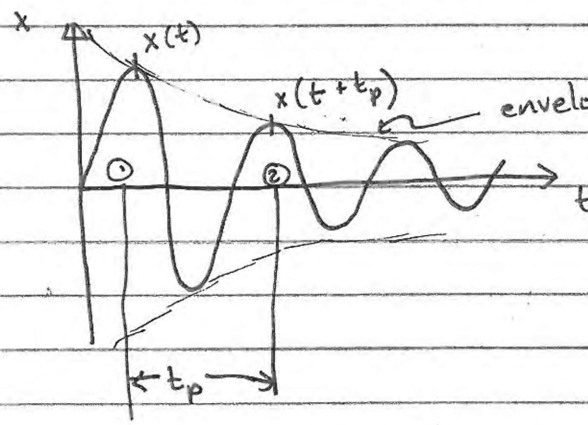
- if the system is damped, then

$t_p = \frac{2\pi}{\omega_d}$ ← period of the damped nat. frequency

- this is nice, but we still don't know what ξ is, let's find it experimentally

define the decrement

$$\sigma = \ln \frac{x(t)}{x(t+t_p)} \quad \Big\} \text{ from experiment}$$



$\Rightarrow e^{-\xi \omega t} = e^{-\xi \omega_n t}$

@ ① $x(t) = Ae^{-\xi \omega t} \sin(\omega_d t + \phi)$

@ ② $x(t+t_p) = Ae^{-\xi \omega_n (t+t_p)} \sin((\omega_d t + t_p) + \phi)$

$$\sigma = \ln \frac{Ae^{-\xi \omega_n t} \sin(\omega_d t + \phi)}{Ae^{-\xi \omega_n (t+t_p)} \sin(\omega_d (t+t_p) + \phi)}$$

$\omega_d t_p = 2\pi$

$\therefore \sin(\omega_d t + \phi) = \sin(\omega_d (t) + 2\pi + \phi)$

$\therefore \sigma = \ln e^{\xi \omega_n t_p} = \xi \omega_n \frac{2\pi}{\omega_d \sqrt{1-\xi^2}}$

- solving for ξ yields

$$\xi = \frac{\sigma}{\sqrt{4\pi^2+\sigma^2}}$$ $\sigma \Rightarrow$ any two succesive peaks

$\xi \Rightarrow$ % critical damping

$$c_{cr} = 2\sqrt{km} \qquad c = \xi c_{cr}$$

- process of identifying $[\omega_n, \xi]$ over many modes is called <u>modal analysis</u>

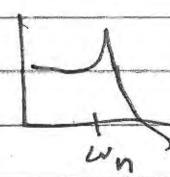
mode shapes $\rightarrow \underline{u}$

- for n peaks

$$\sigma = \frac{1}{n} \ln\left(\frac{x(t)}{x(t+nt_p)}\right)$$

- see (read!) examples 2.13, 2.14

$$|\bar{x}| = \frac{\bar{f}}{m}\left(\frac{1}{\sqrt{\omega_n^2-\omega^2+2\xi\omega_n\omega)^2}}\right) \sim g(\omega)$$

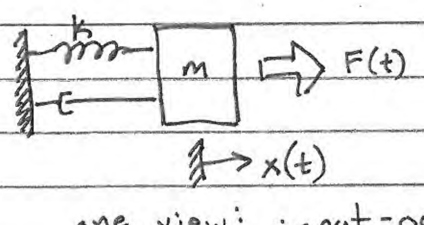


$$|g(\omega)| = \frac{1}{m\sqrt{(\omega_n^2-\omega^2)^2+(2\xi\omega_n\omega)^2}}$$

$$g(\Omega) = \frac{1}{k\sqrt{(1-\Omega^2)^2+(2\xi\Omega)^2}} \qquad \Omega = \frac{\omega}{\omega_n}$$

10/4/13

input-output: linear systems



$$m\ddot{x} + c\dot{x} + kx = F(t)$$

<u>one view:</u> input-output system

input $F(t) \longrightarrow$ [System: e.g. $m\ddot{x} + c\dot{x} + kx = F(t)$] $\longrightarrow x(t)$ output

$\uparrow$ system

<u>Linear system:</u>

If $F_1(t) \Rightarrow x_1(t)$ & $F_2(t) \Rightarrow x_2(t)$,

Then $c_1 F_1 + c_2 F_2 \Rightarrow c_1 x_1 + c_2 x_2$

for all F_1 & F_2

"Linear combination of known inputs $\Rightarrow$ output is same linear combination of corresponding outputs"

For 2-D + 3D mass/spring/dashpot sytems $\Rightarrow$ linear system for small motion (geometric nonlinearities)

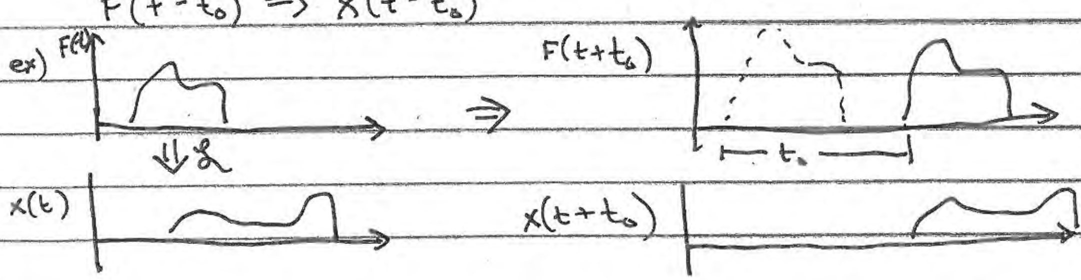
For 1D m/s/d systems $\Rightarrow$ always linear

"Time invariant" = "autonomous" = "time independent"

For all $F(t)$ w/ $F(t) \to x(t)$

$F(t - t_0) \Rightarrow x(t - t_0)$

ex)



Common set of engineering systems are linear, stable, time invariant

<u>Amazing Property</u>

sinewave in $F(t)$ $\rightarrow \boxed{\mathcal{L}} \rightarrow$ sinewave

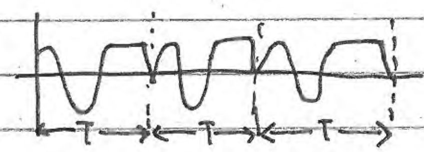
diff phase
diff amp
same freq & shape

<u>Fourrier Series</u>

Any periodic function $F(t)$

$$F(t) = A_0 + \sum_1 A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$$

$F(t) =$



Already know

$$\cos(\omega_n t) \Rightarrow F_{cn} \cos(\omega_n t) + F_{sn} \sin(\omega_n t)$$

$$\sin(\omega_n t) \Rightarrow F_{cn} \cos(\omega_n t) + F_{sn} \sin(\omega_n t)$$

$$F(t) = A_0 + \sum A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$$

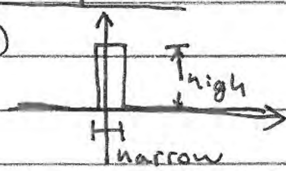
Output $X_0 F_0 + \sum_1 A_n \left[X^c_{cn} \cos(\omega_n t) + X^c_{sn} \sin(\omega_n t)\right] + B_n\left[X^s_{cn} \cos(\omega_n t) + X^s_{sn} \sin(\omega_n t)\right]$

sin response is not ind of cosine response

Same ideas apply w/ Fourrier Transform (in Tongue)

<u>Impulse response</u>

$\delta(t)$



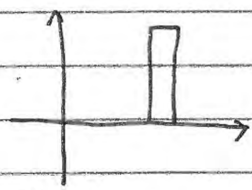
Area = 1

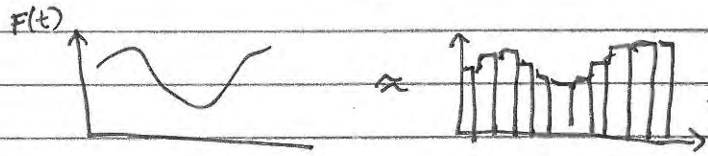
highness $\to \infty$

narrowness $\to 0$

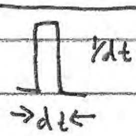
$\delta(t - t')$

↑ const





$$F(t) = \int_{-\infty}^{\infty} F(t')\, \delta(t-t')\, dt'$$



A = 1

$\delta(t) \to \boxed{\begin{matrix}\mathcal{L} \\ \text{T.I.}\end{matrix}} \to h(t)$

$\delta(t-t') \to \boxed{} \to h(t-t')$

$$\underbrace{x(t) = \int_{-\infty}^{\infty} F(t')\, h(t-t')\, dt'}_{\text{convolution integral}}$$

$\boxed{10/7/13}$

Multi-DOF (first of several)

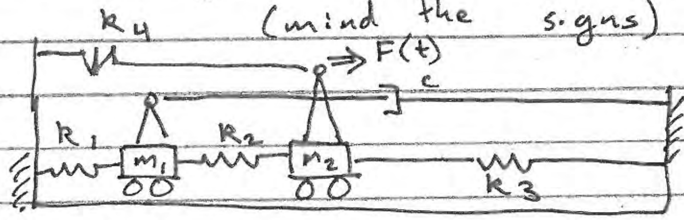
[Ch 4 of Tongue]

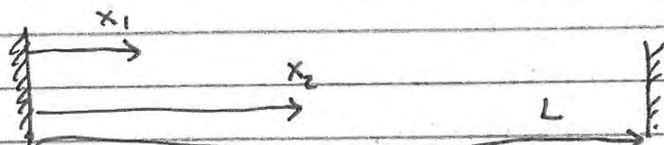
Goal: understand solns of

$$m\ddot{\vec{x}} + c\dot{\vec{x}} + k\vec{x} = F(t)$$

ex): springs, masses, dashpots

(mind the signs)





neglect width of masses

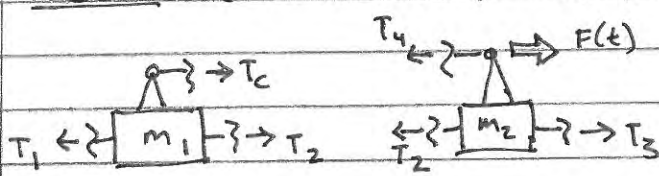
Goal: write eqns

solve a) computer

b) analytical

understand

FBDs: (Tension is positive)



LMB: $\sum F = ma$

$\{ \sum \vec{F} = m\vec{a} \} \hat{i}$

m_1: $T_2 + T_c - T_1 = m_1\ddot{x}_1$

m_2: $T_3 - T_2 - T_4 + F(t) = m_2\ddot{x}_2$

$$T_1 = k_1(x_1 - l_1)$$
$$T_2 = k_2(x_2 - x_1 - l_2)$$
$$T_3 = \cancel{k_3(x_2 - L - l_3)} = T_3 = k_3(L - x_2 - l_3)$$
$$T_4 = k_4(x_2 - l_4)$$
$$T_c = -c\dot{x}_1$$

Substitute

$$m_1 \Rightarrow k_2(x_2 - x_1 - l_2) + -c\dot{x}_1 - k_1(x_1 - l_1) = m_1\ddot{x}_1$$
$$m_2 \Rightarrow k_3(L - x_2 - l_3) - k_2(x_2 - x_1 - l_2) - k_4(x_2 - l_4) + F(t) = m_2\ddot{x}_2$$

write in matrix form

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{m}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}}_{c}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_2+k_3+k_4) \end{bmatrix}}_{k}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -k_2 l_2 + \\ \; \end{bmatrix}}_{F(t)} k_1 l_1$$

$$\begin{bmatrix} -k_2 l_2 + k_1 l_1 \\ +k_2 l_2 + k_4 l_4 - k_3 l_3 + k_3 L + F(t) \end{bmatrix}$$

$$[m]\ddot{\vec{x}} + c\dot{\vec{x}} + K\vec{x} = \vec{B} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix}$$

Equilibrium (soln): $F(t) = 0$, steady state

$$K\vec{x} = \vec{B} \Rightarrow \vec{x}_{ss} = \text{"}K\backslash\vec{b}\text{"}$$

← MATLAB function

Define $\vec{y} = \vec{x} - \vec{x}_{ss}$

$$m\ddot{\vec{y}} + c\dot{\vec{y}} + k\vec{y} = \vec{F}(t)$$

Simple Case: No damping $c=0$

No forcing $F=0$

$$\Rightarrow \underset{\uparrow \text{given}}{M}\ddot{\vec{x}} + \underset{\uparrow}{k}\vec{x} = \vec{0}$$

Method 1: ODE 45, etc.

Method 2: Guess solution

guess: $\vec{x}(t) = \underset{\uparrow \text{const vector}}{\vec{x}}\, e^{i\omega t}$

$$M\left(\ddot{\vec{\underline{x}}} e^{i\omega t}\right) + \underline{K}\vec{x}e^{i\omega t} = \vec{0} \qquad e^{i\omega t} \neq 0$$

$$-\omega^2 M\vec{\underline{x}} + \underline{K}\,\vec{\underline{x}} = \vec{0}$$

$$\underbrace{\left(\underline{K} - \omega^2 M\right)}_{A}\vec{x} = \vec{0} \qquad A\vec{\underline{x}} = 0$$

reduce DE problem to linear algebra problem

`>> A\0`

$$\Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

random matrix
99.99% invertible

- To get nonzero solns, A must be singular
- choose ω such that A is singular

A must be singular $\Leftrightarrow$

$\vec{\underline{x}} \neq \vec{0}$ solns

$\Leftrightarrow \det(A) = 0$

$\Leftrightarrow \det(-\omega^2 M + \underline{K}) = 0$

$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$ roots of "characteristic" polynomial

quadratic formula in ω^2

$\omega_1 = +\sqrt{\quad}$

$$\Rightarrow \quad \vec{x}(t) = c_1 e^{i\omega_1 t}\vec{\overline{X}}_1 + c_2 e^{i\omega_2 t}\vec{\overline{X}}_2$$

Mult: DOF con't | 10/9/13

Recall:

Spring & masses $\Rightarrow m\ddot{\vec{x}} + k\vec{x} = \vec{0}$ ①

ICs $\Rightarrow \underbrace{\vec{x}(0) = \vec{x}_0 , \dot{\vec{x}}(0) = \vec{v}_0}_{\text{constraints}}$ ②

How to solve ① & ② "IVP" initial value problem

Algorithm

① Guess $\vec{x} = \bar{x} e^{i\omega t}$

$\bar{x}$ ← const.

② plug into ①

$$\underbrace{(-\omega^2 M + K)}_{A} \bar{x} = \vec{0}$$

$\Rightarrow \det(A) = 0$

$\Rightarrow$ polynomial in ω^2 e.g. for 2×2 $\Rightarrow$ quadratic eqn in ω^2

$\omega_1 = +\sqrt{\omega_1^2}$

$\omega_2 = +\sqrt{\omega_2^2}$ $\Rightarrow$ positive real roots

$\vdots$

③ For ω_1 we solve

$$(-\omega_1^2 M + k)\bar{x} = 0 \quad ③$$

$\Rightarrow \bar{x}_1$

$\Rightarrow \bar{x}_2$

④ General Solution

$$\vec{x}(t) = c_1 \bar{x}_1 e^{i\omega_1 t} + c_2 \bar{x}_2 e^{i\omega_2 t} + \dots$$

Real soln

$$④ \quad = \bar{x}_1 \left[c_{1c}\cos(\omega_1 t) + c_{1s}\sin(\omega_1 t)\right] + \bar{x}_2 \left[c_{2c}\cos(\omega_2 t) + c_{2s}\sin(\omega_2 t)\right]$$

⑤ IVP

④ $\Rightarrow \vec{x}(0)\ ,\ \dot{\vec{x}}(0)$

set $\vec{x}(0) = \vec{x}_0$

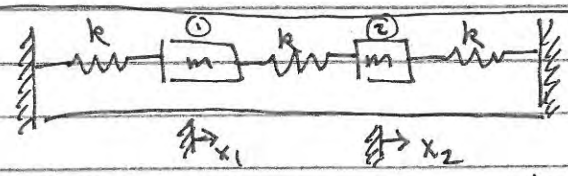
$\dot{\vec{x}}(0) = \vec{v}_0$

$\Rightarrow$ 4 eqns for $C_{1s}, C_{1c}, C_{2s}, C_{2c}$

e.g.

---

ex)



$\underline{LMB_1}$ $\quad m\ddot{x}_1 + kx_1 + -k(x_2 - x_1) = 0$

$\underline{LMB_2}$ $\quad m\ddot{x}_2 + kx_2 + k(x_2 - x_1) = 0$

$$\overbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}^{M} \ddot{\vec{x}} + \overbrace{\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}}^{K} \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

① Guess $\vec{x} = \bar{x}e^{i\omega t} \Rightarrow \det(-\omega^2 M + K) = 0$

$$\det \begin{bmatrix} 2K - \omega^2 M & -K \\ -K & 2K - \omega^2 M \end{bmatrix} = 0$$

$$(2k - \omega^2 m)^2 - k^2 = 0$$

$$4k^2 - 4k\omega^2 m + \omega^4 m^2 - k^2 = 0$$

$$m^2(\omega^2)^2 - 4km\,\omega^2 + 3k^2 = 0$$

$$\omega^2 = \frac{4km \pm \sqrt{16k^2m^2 - 12m^2k^2}}{2m^2}$$

$$\omega^2 = \frac{k}{m}(2 \pm 1)$$

$$\omega_1 = \sqrt{k/m \; 3}$$

$$\omega_2 = \sqrt{k/m \; 1}$$

Solve eqn ③:

$$\omega_2^2 = \frac{k}{m}$$

$$\Rightarrow \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} \bar{x}_{11} \\ \bar{x}_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \bar{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\omega_1^2 = 3\frac{k}{m}$$

$$\Rightarrow \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{bmatrix} \bar{x}_{21} \\ \bar{x}_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \bar{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Gen solution:

$$\vec{x}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(c_{1c} \cos\sqrt{\frac{k}{m}}\, t + c_{1s} \sin\sqrt{\frac{k}{m}}\, t \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(c_{2c} \cos\sqrt{\frac{3k}{m}}\, t + c_{2s} \sin\sqrt{\frac{3k}{m}} \right)$$

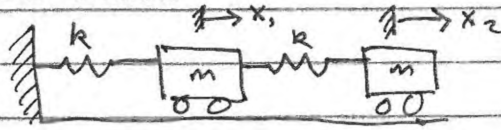
10/11/13

Multi-DOF con't

Vibration absorption

Normal modes

ex)



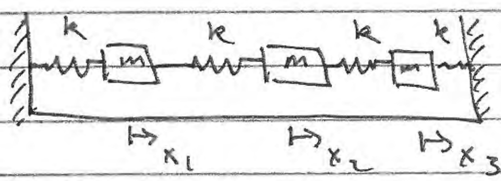
<u>method 1</u>

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Guess $\vec{x} = \underset{\uparrow \text{ const}}{\bar{X}} e^{i\omega t}$

$$\Rightarrow \left[-\omega^2 M + k\right] \bar{X} = \vec{0}$$

ex)



$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Guess: $\bar{X} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\omega_1 = \sqrt{\frac{2k}{m}}$ $\quad \bar{X} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Guess 2: $\bar{X} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ Does not work because middle mass can't move without force, can't have force if $(x_3 - x_2) = (x_2 - x_1) = 0$

Guess: $\begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix}$

check: $\left.\left(\frac{k}{m}\right)_{eff}\right|_1 = \left.\left(\frac{k}{m}\right)_{eff}\right|_2$

$$\frac{F_1/d_1}{m_1} \stackrel{?}{=} \frac{F_2/d_2}{m_2}$$

$$\frac{(2k - ak)/1}{m_1} = \frac{2k(a-1)/a}{m_2}$$

$$\frac{\cancel{k}(2-a)}{\cancel{m_1}} = \frac{\cancel{2k}(a-1)}{a\,\cancel{m_2}}$$

$$(2-a)\,a = (a-1)\,2$$

$$2a - a^2 = 2a - 2 \qquad \cancel{a^2 - 4a + 2a + 2 = 0}$$

$$0 = a^2 - 2 \qquad \cancel{(a-1)^2 = 0}$$

$$\boxed{a = \pm\sqrt{2}} \qquad \cancel{a = 4}$$

$$\Rightarrow \bar{x}_2 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \quad \bar{x}_3 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \qquad \cancel{\omega_1 = \sqrt{\tfrac{k}{m}},\ \omega_2 = \sqrt{\tfrac{k}{m}}}$$

$$\omega_1 = \sqrt{\frac{k}{m}(2-\sqrt{2})}$$

$$\omega_2 = \sqrt{\frac{k}{m}(2+\sqrt{2})}$$

Add in forcing

$$m\ddot{\vec{x}} + k\vec{x} = \vec{F}_0 \sin(\omega t)$$

e.g. $\vec{F} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\vec{x}_h(t \to \infty) = \vec{0}$ homogenous dies away

A particular soln

guess: $\vec{x}(t) = \bar{x} \sin(\omega t)$

$$M\ddot{\vec{x}} + k\vec{x} = \vec{F}_0 \sin(\omega t)$$

$$\underbrace{(-\omega^2 M + K)}_{A}\,\bar{x} = \bar{F}_0$$

Assuming $\omega \neq \omega_i \leftarrow$ modal solutions

$$\bar{x} = (-\omega^2 M + K)^{-1} F_0$$

$\Rightarrow$ force a system w/ $\sin(\omega t)$, eventually it shakes sinusoidally

10/16/13

Lagrange Eqs:

$$\mathcal{L} = E_k - E_p$$

for all: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q} = 0$

conservative holonomic time ind.

Near equilibrium: $\vec{q} = \vec{0}$ $\qquad \ddot{q}\big|_{\dot{q}=0} = 0$

Stable equilibrium: $E_p > 0 \quad \vec{q} \neq \vec{0}$

potential energy minimum: $E_p = 0 \quad \vec{q} = \vec{0}$

Kinetic energy always positive:

$$E_k = \frac{\sum m_i v_i^2}{2}$$

Near equilibrium

$$E_k = \frac{1}{2}\underbrace{\left[\sum\sum \frac{\partial E_k}{\partial \dot{q}_i \partial \dot{q}_j}\right]_{\dot{q}_i = 0}}_{M_{ij}} \dot{q}_i + \dot{q}_j + \ldots$$

symmetric positive definite $\qquad M_{ij}\dot{q}_i\dot{q}_j > 0$

ex) cx^2 positive definite if $c > 0$ for all $\vec{q} \neq \vec{0}$

ASIDE

General Taylor Series $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$

$$f(\vec{x}) = f(\vec{0}) + \sum \frac{\partial f}{\partial x_i}\bigg|_{\vec{x}=\vec{0}} x_i + \frac{1}{2}\sum\sum \frac{\partial^2 f}{\partial x_i x_j} x_i x_j + \sum\sum\sum \ldots$$

for potential energy: $f(\vec{0})$ does not matter, can add constant; second-order term: only important term; $\sum\sum\sum$: negligible

Lagrange Eqs

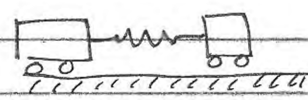
$$\mathcal{L} = E_K - E_p = \frac{1}{2}\sum\sum M_{ij}\,\dot{q}_i\dot{q}_j + \frac{1}{2}\sum\sum K_{ij}\,q_i q_j$$

$$\Rightarrow M\ddot{\vec{q}} + K\vec{q} = \vec{0}$$

symmetric and positive definite

Actually all we need is K positive semi definite

e.g.



How to solve $M\ddot{\vec{x}} + K\vec{x} = 0$

with $\vec{x}(0) = \vec{x}_0$, $\vec{v}(0) = \vec{v}_0$

Method 1:

$z = [x;\ v]$;

$z_0 = [x_0;\ v_0]$;

$\dot{x} = v$;

$\dot{v} = -M \backslash (K * x)$;

Method 2: guess $\vec{x} = \bar{x}\cos(\omega t)$

$$\Rightarrow \underbrace{\left(-\omega^2 M + K\right)}_{\det(\ )\,=\,0}\bar{x} = \vec{0}$$

polynomial $\Rightarrow$ solve

$\Rightarrow$ find $\bar{x}$

$\Rightarrow$ add up solns

$\vec{x} = \eta_i\,\bar{x}_i\cos(\omega_i t)$ + sin terms like this

constants

$M + K \Rightarrow \omega_i,\ \bar{x}_i$

$I \Rightarrow \eta_{ci}$ + η_{si}

Method 2b) Short Cut

$$[V, D] = \text{eig}(K, M)$$

$D \to \begin{bmatrix} \omega_1^2 & & 0 \\ & \omega_2^2 & \\ 0 & & \ddots \end{bmatrix}$

$V \to [\bar{x}_1 \mid \bar{x}_2 \mid \cdots]$

Method 2c

$$M^{-1} \cdot \{ M\ddot{\vec{x}} + K\vec{x} = \vec{0} \}$$

$$\ddot{\vec{x}} + \underbrace{M^{-1}K}_{\text{nonsingular matrix}} \vec{x} = \vec{0}$$

M^{-1}, K are symmetric $\Rightarrow$ product may or may not be symmetric, usually not

$$\Rightarrow [V, D] = \text{eig}(M^{-1}K)$$

$D \to \begin{bmatrix} \omega_1^2 & & 0 \\ 0 & \omega_2^2 & \\ & & \ddots \end{bmatrix}$

$V \to [\vec{x}_1 \mid \vec{x}_2 \mid \cdots \vec{x}_n]$

works but we don't have confidence because $M^{-1}K$ generally not symmetric

insecure: maybe not enough eigenvectors to span set of initial conditions

10/18/13

Today

$$M\ddot{\vec{x}} + K\vec{x} = \vec{0}$$

$$\Rightarrow \ddot{\vec{x}} + M^{-1}K\vec{x} = 0$$

guess $\vec{x} = \underbrace{\bar{x} \cos\omega t}_{\text{consts}}$

$$\Rightarrow -\omega^2 \bar{X} + M^{-1}K\bar{X} = 0$$

$$\Rightarrow M^{-1}K\bar{x} = \omega^2 \bar{x}$$

$$[A\bar{x} = \lambda\bar{x}]$$

standard e-vector, e-value problem

But! A generally not symmetric

$\Rightarrow$ ① can't be sure there are n evectors

② e-vectors not mutually $\perp$

Slightly diff approach:

[change of coords before guessing soln]

Define $U = [\underbrace{\bar{x}_1 \mid \bar{x}_2 \mid \bar{x}_3 \mid \cdots \mid \bar{x}_n}_{\text{e-vectors of } M^{-1}K}]$

$$\vec{x} = \bar{x}_1 \zeta_1(t) + \bar{x}_2 \zeta_2(t) + \dots$$

new coords

$$\vec{H} = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \\ \vdots \end{bmatrix} \Rightarrow \vec{x} = \underbrace{U\vec{H}}$$

$\hookrightarrow$ linear combination of columns of the matrix

EOM: $\Rightarrow \{U\ddot{\vec{H}} + \underbrace{M^{-1}KU}_{U\Lambda}\vec{H} = 0\}$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \ldots \end{bmatrix}$$

$U^{-1}\{\} \Rightarrow \ddot{\vec{H}} + \Lambda\vec{H} = \vec{0}$

$U\Lambda\vec{H}$

comp by comp

$$\begin{bmatrix} \ddot{\eta}_1 + \lambda_1 \eta_1 = 0 \\ \ddot{\eta}_2 + \lambda_2 \eta_2 = 0 \\ \vdots \end{bmatrix}$$

decoupled harmonic oscillation equations

modal coordinates

Aside: [V, D] = eig(A)

>> A * V - V * D

$$\begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ & \vdots & & \ddots \end{bmatrix}$$

$\Rightarrow \eta_1 = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t)$

$\eta_2 = A_2 \cdots\cdots$

$\vdots$

$$\Rightarrow \vec{x} = U \cdot \vec{H} = [\bar{x}_1 | \bar{x}_2 \cdots] \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \end{bmatrix}$$

Official Normal Modes Method

$M\ddot{\vec{x}} + K\vec{x} = \vec{0}$

Assume $\vec{q} = M^{1/2}\vec{x}$

$\vec{x} = M^{-1/2}\vec{q}$

M positive definite and symmetric

$\Rightarrow M^{1/2}$ is sym, positive def exists

☆☆☆ * symmetric real matrix has real eigenvalues and eigenvectors and they are orthogonal ☆

Facts to remember:

A is symmetric $\Rightarrow$ has n e-vectors

mutually orthogonal

λ are real

A positive definite $\Rightarrow \lambda_i > 0$

$$\{ M M^{-1/2} \ddot{\vec{q}} + K M^{-1/2} \vec{q} = 0 \}$$

$$M^{-1/2}\{\} \Rightarrow \underbrace{M^{-1/2} M M^{-1/2}}_{M^{1/2} \cdot M^{1/2}} \ddot{\vec{q}} + M^{-1/2} K M^{-1/2} \vec{q} = 0$$

$$\Rightarrow \quad \ddot{\vec{q}} + \overbrace{M^{-1/2} K M^{-1/2}}^{\tilde{K}} \vec{q} = \vec{0}$$

$\tilde{K}$ is symmetric positive definite

<u>FIND</u> e-values & r vectors of $\tilde{K}$

$$\Rightarrow \quad \underbrace{P = [\bar{q}_1 \mid \bar{q}_2 \ \cdots]}_{\text{e-vectors of } \tilde{K}} \qquad \underbrace{\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}}_{\text{e-values of } \tilde{K}}$$

Another change of variables

$$\vec{q} = P\vec{r} = \bar{q}_1 r_1(t) + \bar{q}_2 r_2(t) + \bar{q}_3 r_3(t) \ldots\ldots$$

modal coords

e-vectors of $\tilde{K} = M^{-1/2} K M^{-1/2}$

$$P\ddot{\vec{r}} + \tilde{K} P \vec{r} = \vec{0}$$

$$P\ddot{\vec{r}} + \tilde{K} P \vec{r} = \vec{0}$$

$$\{ P\ddot{\vec{r}} + P\Lambda \vec{r} = 0 \}$$

$$P^{-1}\{\} \quad \ddot{\vec{r}} + \Lambda \vec{r} = 0$$

diagonal

$$\left.\begin{aligned} \ddot{r}_1 + \lambda_1 r_1 = 0 \\ \ddot{r}_2 + \lambda_2 r_2 = 0 \end{aligned}\right\} \text{decoupled harmonic oscillators}$$

$\Rightarrow \vec{r}(t)$

$$\underbrace{P\vec{r}(t)}_{\vec{q}} \qquad \vec{x}(t) = M^{-1/2} P \vec{r}(t)$$

NOTE $\bar{q}_1, \bar{q}_2, \cdots$ are mutually orthogonal

10/21/13

GOAL: $M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{F}(t)$

Look at simple comparison problem

$M\ddot{\vec{x}} + K\vec{x} = \vec{0}$ ★

$\Rightarrow$ Normal modes (superposition)

Forcing w/ no damping

$M\ddot{\vec{x}} + K\vec{x} = \vec{F}(t)$

$\hookrightarrow \vec{F}_0 \sin \omega t + \vec{F}_c$

$\vec{F}_c \Rightarrow$ const soln

$\vec{F} = \vec{0} \Rightarrow$ ★

$\vec{F} = \bar{F}_0 \sin \omega t$

If $\omega \neq \omega_i$

← normal mode freq

$\Rightarrow \vec{x}(t) = \bar{x} \sin \omega t$

$\bar{x} = (-\omega^2 M + K)^{-1} \vec{F}_0$

To do:

1) Damping w/ no forcing

2) Damping w/ forcing

DAMPING:

$M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{0}$ (★★)

Might guess:

$\vec{x}(t) = \bar{X} e^{-\lambda t}(A\cos(\omega t) + B\sin(\omega t))$

Wishful thinking! Does not work out like that (always)

First order form

Define $\vec{v} = \dot{\vec{x}}$ (1)

(★★) $\Rightarrow M\dot{\vec{v}} + C\vec{v} + K\vec{x} = \vec{0}$

$\dot{\vec{v}} = -(M^{-1}C\vec{v} + M^{-1}K\vec{x})$ (2)

(1) + (2) are 2n first order ODEs

2n

$2n \times 1$

$A: 2n \times 2n$

$$\begin{bmatrix} \dot{\vec{x}} \\ \dot{\vec{v}} \end{bmatrix} = \begin{bmatrix} [0]_{n\times n} & [I]_{n\times n} \\ [-M^{-1}K] & [-M^{-1}C] \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{v} \end{bmatrix}$$

$$\vec{z} = \begin{bmatrix} \vec{x} \\ \vec{v} \end{bmatrix} \qquad \dot{\vec{z}} = A\vec{z} \quad (\star\star\star)$$

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \neq \text{symmetric}$$

has complex e-values + complex e-vectors
unless $C = [0]$

To solve $(\star\star\star)$

guess $\vec{z}(t) = \bar{z}e^{\lambda t} \Rightarrow \frac{\partial}{\partial t}\left(\bar{z}e^{\lambda t}\right) = A\bar{z}e^{\lambda t}$

$$\left.\begin{aligned} &\lambda\bar{z}e^{\lambda t} = A\bar{z}e^{\lambda t} \end{aligned}\right\} \text{in terms of } z$$

$$\lambda\bar{z}e^{\lambda t} = A\bar{z}e^{\lambda t}$$

$$A\bar{z} = \lambda\bar{z}$$

$$[V, D] = \text{eig}(A)$$

$$V = [\vec{v}_1 | \vec{v}_2 | \cdots] \qquad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{bmatrix}$$

$$\vec{z}(t) = c_1\bar{z}_1 e^{\lambda_1 t} + c_2\vec{z}_2 e^{\lambda_2 t} + \ldots\ldots c_n\vec{z}_n e^{\lambda_n t}$$

NOTE: $\bar{z}_1$ is complex
λ_1 is complex

2 ways to deal with complex solutions:

1) Pick complex c_i to kill imaginary parts
2) Take real part and separately Imaginary part of soln

Let's look at 1^{st} mode

$$\vec{z}(t) = e^{(\lambda_r t + i\omega t)} \bar{z}$$

$$= e^{(\lambda_r t)} (\cos \omega t + i \sin(\omega t)) \left[\bar{z}_r + i \bar{z}_i \right]$$

$$\mathbb{R}\left(\vec{z}(t)\right) = e^{\lambda_r t} \left[\cos(\omega t)\, \bar{z}_r - \sin \omega t\, \bar{z}_i \right]$$

$\bar{z}_r \neq \bar{z}_i$

$\Rightarrow$ a single damped oscillator

$Re(\lambda) = \lambda_r$

$Im(\lambda) = \omega$

ASIDE

Freshman Calculus

$\dot{x} = a x$ scalar eqn

one way to solve: Assume soln has Taylor series

$$x = c_0 + c_1 t + c_2 t^2 + \ldots$$

$$c_1 = \dot{x}\big|_0$$

$$c_2 = \tfrac{1}{2} \ddot{x}\big|_0$$

$$c_3 = \tfrac{1}{3!} \dddot{x}\big|_{t=0}$$

$\boxed{10/23/13}$ Calculus Review

Scalar eqn.

$\dot{x} = ax$ w/ $x(0) = x_0$

soln is $x(t)$

$$\dot{x} = ax$$

$$\ddot{x} = \frac{\partial}{\partial t} ax = a\dot{x} = a(ax) = a^2 x$$

$$\dddot{x} = \frac{\partial}{\partial t}(\ddot{x}) = a^2 \dot{x} = a^2(ax) = a^3 x$$

if: given 1^{st} order ODE

& given soln at one t

$\Rightarrow$ Taylor series for soln at "all" times

Taylor series at $x=0$

$$x(t) = x_0 t + \dot{x} t + \frac{1}{2}\ddot{x} t^2 + \frac{1}{3!}\dddot{x} t^3 + \dots$$

$$x(t) = x_0 + a x_0 t + \frac{a^2 x_0 t^2}{2} + \frac{a^3 x_0 t^3}{3!} \dots$$

$$= x_0 \left(1 + at + \frac{a^2t^2}{2} + \frac{a^3t^3}{3!} + \dots\right)$$

$$x(t) = x_0 e^{at}$$

$$\frac{d}{\delta t} x(t) = 0 + a + a^2 t + \frac{3a^3t^2}{2\cdot 3} + \dots$$

$$= a\, x(t)$$

$$\boxed{\begin{array}{l} \dot{x} = ax \quad w/ \; x(0) = x_0 \\ \text{has soln } x = x_0 e^{at} \end{array}}$$

Back to multivar world

$$M\ddot{\vec{x}} + c\dot{\vec{x}} + K\vec{x} = \vec{0}$$

$$\dot{\vec{z}} = A\vec{z} \qquad w/ \; \vec{z}(0) = \vec{z}_0$$

$$\vec{z} = \begin{bmatrix} \vec{x} \\ \vec{v} \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I \\ -M^{-1}c & -M^{-1}K \end{bmatrix}$$

Guess Taylor series soln.

$$\vec{z}(t) = \vec{z}_0 \left(1 + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \dots\right)$$

Check: $\dot{\vec{z}} = A\vec{z}$

$$\hookrightarrow \vec{z}_0 \left(I + At + \frac{A^2t^2}{2} + \dots\right)$$

$$\frac{d}{\delta t}\left[\vec{z}_0 \left(I + At + \frac{A^2t^2}{2} + \dots\right)\right]$$

define $e^{At} = \left(I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \dots\right)$

Soln of $\dot{\vec{z}} = A\vec{z}$ w/ $\vec{z}(0) = \vec{z}_0$

$$\boxed{\cancel{\vec{z} = \vec{z}_0 e^{At}}}$$

$$\boxed{\vec{z} = e^{At}\vec{z}_0}$$

Why better than

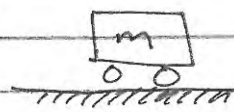
$$\vec{z} = c_1 e^{\lambda_1 t}\vec{z}_1 + c_2 e^{\lambda_2 t}\vec{z}_2 + \ldots$$

$$\vec{z}_i, \lambda_i = \text{e - things of } A$$

① Tidy

② Don't have problem of missing e-vectors

ex)



1) old method

$$m\ddot{x} + \overset{0}{\cancel{c}}\dot{x} + \overset{0}{\cancel{k}}x = 0$$

guess: $\vec{z} = \begin{bmatrix} x \\ v \end{bmatrix} = \bar{z}e^{\lambda t}$

$$\dot{\vec{z}} = \overbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{A}\vec{z}$$

$$\det(A - \lambda I) = 0$$

$$\det\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = \lambda^2 = 0$$

Look for e-vector $\bar{\omega}$ for $\lambda_1 = 0$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \bar{z}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{z}(t) = c_1 \overset{1}{\cancel{e^{0t}}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + ?$$

$$= c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + ?$$

trick ______ $\cancel{c_2 t\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$ $\quad c_2\left(\begin{bmatrix} t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$$\begin{bmatrix}x\\v\end{bmatrix} = \begin{bmatrix}x_0\\v_0\end{bmatrix} + \begin{bmatrix}v_0 t\\0\end{bmatrix}$$

Try matrix exponential soln

$$\vec{z} = \cancel{\begin{bmatrix}x_0\\v_0\end{bmatrix} e^{At}} \qquad e^{At}\begin{bmatrix}x_0\\v_0\end{bmatrix}$$

$$= \cancel{\begin{bmatrix}x_0\\v_0\end{bmatrix}\left(I + At + A^2t^2\right.}$$

$$= \left(I + At + \frac{A^2t^2}{2!}\right)\begin{bmatrix}x_0\\v_0\end{bmatrix}$$

$$= \left(I + \begin{bmatrix}0&1\\0&0\end{bmatrix}t + 0 + 0\right)\begin{bmatrix}x_0\\v_0\end{bmatrix}$$

$$\begin{bmatrix}x\\v\end{bmatrix} = \begin{bmatrix}x_0\\v_0\end{bmatrix} + \begin{bmatrix}0&t\\0&0\end{bmatrix}\begin{bmatrix}x_0\\v_0\end{bmatrix}$$

$$= \cancel{\begin{bmatrix}x_0 t\ v_0 t\\ \ \end{bmatrix}} = \begin{bmatrix}x_0 + v_0 t\\v_0\end{bmatrix}$$

$$\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{F}(t)$$

$e^{At} \Rightarrow$ anal solution to homogeneous eqns
(transient soln)

What about particular soln?

$$M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \bar{F}e^{\lambda t}$$

$M, c, K, \vec{F}, \lambda$ given

FIND $\vec{x}(t)$

$\vec{x} = \bar{x}e^{i\omega t}$ good guess unless $c = [0]$ & $\omega = \omega_i$

$$-\omega^2 M\bar{x} + i\omega \overset{\zeta^{c}}{C}\bar{x} + K\bar{x} = \bar{F}$$
$$(-\omega^2 M + i\omega C + K)\,\bar{x} = \bar{F}$$

$$\bar{x} = [-\omega^2 M + i\omega C + K] \backslash \bar{F}$$

$\bar{x}$ is complex, only taking real part

10/25/13 GARCIA

MDOF Systems

- damping

$$M\ddot{\underline{x}} + K\underline{x} = B_f f_i(t)$$

- physical coordinates

$$\underline{x} = M^{-1/2} q$$

and pre-multiplying by $M^{-1/2}$

$$M^{-1/2} M M^{-1/2} \ddot{q} + M^{-1/2} K M^{-1/2} q = M^{-1/2} B_f f(t)$$

$= 0$ homogeneous to get e-values and e-vectors

$$I\ddot{q} + \hat{K} q = 0 \Rightarrow$$

Why change variables?

$M^{-1/2} K M^{-1/2} \Rightarrow$ symm, positive definite, orthogonal eigenvectors

$M^{-1} K \Rightarrow$ non-symetric (symmetric if $m_1 = m_2 = \cdots$)

$$|\hat{K} - \lambda I| = 0 \quad \lambda_1, \lambda_2$$

e-vectors $\underline{v}_1, \underline{v}_2$ are $\perp$

Create a modal matrix

$$P = [\underline{v}_1 \vdots \underline{v}_2]$$

$$P^T P = I \qquad \text{where } \underline{v}_i : \text{ orthonormal} \quad \left.\begin{matrix} \\ \underline{v}_1^T \underline{v}_1 = 1 \\ \underline{v}_1^T \underline{v}_2 = 0 \end{matrix}\right\} \text{not required, but neater}$$

$$M^{-1}K : \underline{u}_1, \underline{u}_2$$

$$\underline{u}_1^T \underline{u}_2 \neq 0$$

$$q = M^{+1/2}\underline{x} \;\Big\}\; q \text{ has no physical meaning}$$

$$\Rightarrow I\ddot{q} + \tilde{K}q = M^{-1/2} B_f\, f(t)$$

$$q = P\underline{r}$$

$$I P \underline{r} + \tilde{K} P \underline{r} = M^{-1/2} B_f\, f(t)$$

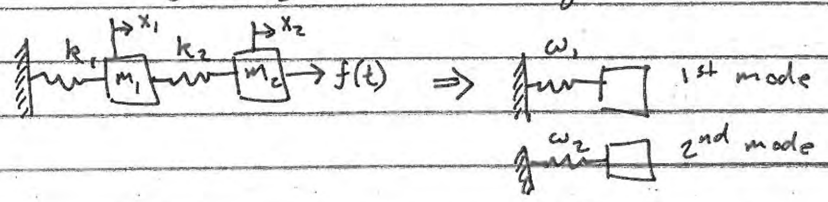
pre multiply by P^T

$$P^T I P \ddot{\underline{r}} + P^T \tilde{K} P\, \underline{r} = P^T M^{-1/2} B_f\, f(t)$$

$$I \ddot{\underline{r}} + \Lambda_K \underline{r} = P^T M^{-1/2} B_f\, f(t)$$

$\underline{r}$ = modal coordinates

$$\underline{r} = \begin{Bmatrix} r_1(t) \\ r_2(t) \\ \vdots \end{Bmatrix}$$ ← transformed to a bunch of 1DOF systems



ex $m_1 = 9$, $m_2 = 1$, $K_1 = 24$, $K_2 = 3$ } try and decouple the system

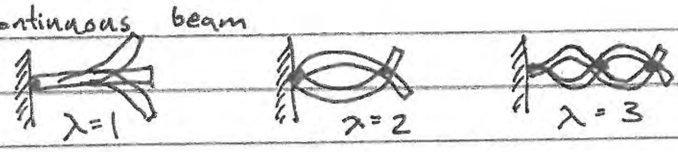
$\omega_1^2 = 2$

$\omega_2^2 = 4$

$$I\ddot{\underline{r}} + \Lambda_K \underline{r} = \Gamma_i f(t)$$

modal participation factors

continuous beam



<u>Damping</u> in MDOF systems

$$M\ddot{\underline{x}} + D\dot{\underline{x}} + K\underline{x} = B_f f(t)$$

where $D = \alpha M + \beta K$

α, β = chosen and allow a form of damping that permits us to decouple the system

$$\left.\begin{aligned} \underline{x} &= M^{-1/2}\underline{q} \\ \underline{q} &= P\underline{r} \end{aligned}\right\} \underline{x} = M^{-1/2}P\underline{r}$$

$$MM^{-1/2}P\ddot{\underline{r}} + DM^{-1/2}P\dot{\underline{r}} + KM^{-1/2}P\underline{r} = B_f f(t)$$

- assume we found P from $\hat{K}$
- pre multiply by $P^T M^{-1/2}$

$$P^T M^{-1/2} M M^{-1/2} P\ddot{\underline{r}} + P^T M^{-1/2} D M^{-1/2} P\dot{\underline{r}} + P^T M^{-1/2} K M^{-1/2} P\underline{r} = P^T M^{-1/2} B_f f(t)$$

$$I\ddot{\underline{r}} + \Lambda_D \dot{\underline{r}} + \Lambda_K \underline{r} = \Gamma f(t)$$

$$\Lambda_D = P^T M^{-1/2}(\alpha M + \beta K) M^{-1/2} P$$

$$\Lambda_D = \alpha P^T M^{-1/2} \overset{M}{\;} M^{-1/2} P + \beta P^T M^{-1/2} \overset{K}{\;} M^{-1/2} P$$

$$\Lambda_D = \alpha P^T M^{-1/2} M M^{-1/2} P + \beta P^T M^{-1/2} K M^{-1/2} P$$

$$\Lambda_D = \alpha I + \beta \Lambda_K = 2 \zeta_i \omega_i$$ ← diagonal of this

Normal modes:

- · nodes are fixed
- modes being real numbers, not complex

$\underline{u}_i = m^{-1/2} \underline{v}_i$ have to convert e-vectors

$\underline{x}_i = M^{-1/2} q_i$ to physical coordinates to get modes

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Today | 10/28/13

Normal mode damping (cont'd)

Structural Vibes

Start with

$M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \vec{F}(t)$ & try to decouple the eqns into separate 1-DOF systems

(1) $\vec{x} = M^{-1/2}\vec{q}$

$M^{-1/2} \cdot \{\} \Rightarrow$

$$\ddot{\vec{q}} + M^{-1/2} C M^{-1/2} \dot{\vec{q}} + \underbrace{M^{-1/2} K M^{-1/2}}_{\tilde{K}} \vec{q} = M^{-1/2} \vec{F}(t)$$

P = e-vectors of $\tilde{K}$

normalized

$\vec{q} = P\vec{r}$

↑ normal mode shapes

NOTE: $\vec{K} P = P \Lambda$

NOTE: $P^{-1} = P'$

inv(P)

$P^{-1} * \{\ \} \Rightarrow$

$$\ddot{\vec r} + \underbrace{P' M^{-1/2} C M^{-1/2} P \dot{\vec r}} + \Lambda \vec r = P' M^{-1/2} \vec F$$

$$\underbrace{P' M^{-1/2} C M^{-1/2} P}_{\tilde C \text{ is a mess}} \qquad \Lambda = \begin{bmatrix} \sim & 0 \\ 0 & \sim \end{bmatrix} \text{ e-values of } \hat K$$

$$\ddot{\vec r}_7 + [\tilde C \dot{\vec r}]_7 + \omega_7^2 \vec r_7 = [P' M^{-1/2} \vec F]_{\cancel{z}7} \quad \star$$

$\star$ is decoupled <u>but</u> for damping terms

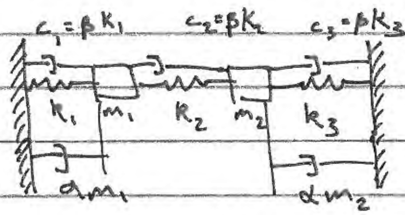
$\Rightarrow$ can't decouple the eqns

$\Rightarrow$ wish the problem away

Method 1: <u>Assume</u> $C = \alpha M + \beta K$

$$\Rightarrow \ddot{\vec r} + (\alpha I + \beta \Lambda) \dot{\vec r} + \Lambda \vec r = \hat{\tilde{F}}$$

$$\underbrace{\ddot{\vec r}_7 + (\alpha + \beta \omega_7^2) \dot{\vec r}_7 + \omega^2_7 \vec r_7 = [\hat{\tilde F}]_7}_{\text{decoupled eqn}}$$

ex) 

add dashpot next to every spring $c_i = \beta k_i$

add dashpot to ground for every mass $c_j = \alpha m_j$

$\star\star$ [β has biggest affect on high freq modes

α has biggest affect on low freq modes]

$\star\star$ [low freq modes occur when masses move together

high freq modes occur when masses move opposite each other]

Method 2

$$P^{-1} M^{-1/2} C M^{-1/2} P = \tilde{C}$$

$$\tilde{C} = \begin{bmatrix} a & \cancel{\ } & \cancel{\ } \\ \cancel{\ } & \ddots & \\ \cancel{\ } & & g \end{bmatrix}$$ cross off off-diag terms

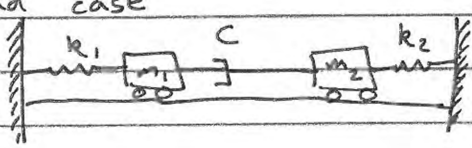
a) given model set

$\tilde{c}_{ij} = 0$ if $i \neq j$

b) Exp in syst with no model

* excite one mode i
* look at its rate of decay
* pick damping for that mode
* call it $\tilde{c}_i$

ex) bad case



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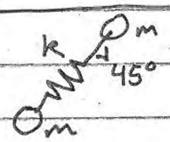
ex)



$\boxed{10/30/13}$

$$P = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$ where $\vec{x} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}$

$\omega^2 = 0$

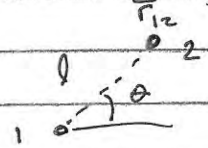
$\hookrightarrow \omega^2 = \frac{2k}{m}$

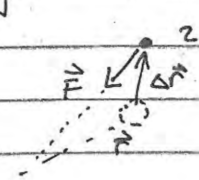
$$M = \begin{bmatrix} m_1 & & & 0 \\ & m_1 & & \\ & & m_2 & \\ 0 & & & m_2 \end{bmatrix} \qquad K = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$|\Delta \vec{r}| \ll \vec{r}$

$\Delta l = \Delta r \cdot \hat{r}$

$\Delta l = \Delta \vec{r} \cdot \frac{\vec{r}}{|\vec{r}|}$





$$\vec{F} = K(\Delta l) \frac{-\vec{r}}{|\vec{r}|}$$

$$\vec{F} = \frac{K \Delta \vec{r} \cdot \vec{r}}{|\vec{r}|^2} \vec{r}$$

$$\vec{F} = \frac{-k\,\Delta\vec{r}\cdot\vec{r}}{|\vec{r}|^2} \qquad \vec{F} = \begin{bmatrix} F_x \\ F_y \end{bmatrix} \qquad \vec{r} = \begin{bmatrix} r_x \\ r_y \end{bmatrix}$$

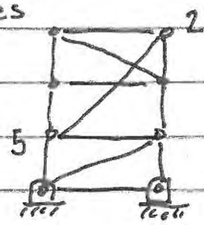
$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \frac{-K}{|\vec{r}|^2} \begin{bmatrix} \overset{r_x r_x}{\cancel{r_x r_y}} & r_x r_y \\ r_y r_x & r_y r_y \end{bmatrix} \begin{matrix} \Delta r_x \\ \Delta r_y \end{matrix}$$

$$= -K \underbrace{\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}}_{-\hat{K}} \begin{bmatrix} \Delta r_x \\ \Delta r_y \end{bmatrix}$$

$$K = \left[\begin{array}{c|c} \hat{K} & -\hat{K} \\ \hline -\hat{K} & \hat{K} \end{array}\right]$$

11/1/13 Structural Vibes

ex) each bar is a spring



focus bar 2-5

What is force $\vec{F}$ on node 2 due to spring/bar 2-5?

For each node 7

$\vec{F}_7 = m_7 \vec{a}_7$

↳ $\sum$ Forces from springs

Look at one spring from node 5 to node 2:

Force on ② from spring: 2-5:

$$\vec{F} = -\Delta l\, k \frac{\vec{r}}{|\vec{r}|} = -k(\Delta l)\,\hat{\lambda}$$

$$= -k\left[|\vec{r}| - |\vec{r}_0|\right] \frac{\vec{r}}{|\vec{r}|}$$

NOTE:
$$\sqrt{1+\epsilon} \approx 1+\frac{\epsilon}{2}$$
$$\frac{1}{1+\epsilon} \approx 1 - \epsilon$$

$$= -k\left[|\vec{r}_0+\Delta\vec{r}| - |\vec{r}_0|\right]\frac{\vec{r}_0+\Delta\vec{r}}{|\vec{r}_0+\Delta\vec{r}|}$$

$$= -k\left[\sqrt{(\vec{r}_0+\Delta\vec{r})\cdot(\vec{r}_0+\Delta\vec{r})} - \sqrt{\vec{r}_0\cdot\vec{r}_0}\right]\frac{\vec{r}_0+\Delta\vec{r}}{\sqrt{(\vec{r}_0+\Delta\vec{r})\cdot(\vec{r}_0+\Delta\vec{r})}}$$

$$= -k\left[\sqrt{\vec{r}_0\cdot\vec{r}_0 + 2\Delta\vec{r}\cdot\vec{r}_0 + \Delta\vec{r}\cdot\Delta\vec{r}} - \sqrt{\vec{r}_0\cdot\vec{r}_0}\right]\frac{\vec{r}_0+\Delta\vec{r}}{\sqrt{\vec{r}_0\cdot\vec{r}_0+2\Delta\vec{r}\cdot\vec{r}_0+\Delta\vec{r}\cdot\Delta\vec{r}}}$$

divide by r_0 $\qquad -1$

$$= -k\sqrt{1+2\frac{\Delta\vec{r}\cdot\vec{r}_0}{|\vec{r}_0|^2}+\frac{\Delta\vec{r}\cdot\Delta\vec{r}}{|\vec{r}_0|^2}}\cdot\frac{\vec{r}+\Delta\vec{r}}{\sqrt{1+2\frac{\Delta\vec{r}\cdot\vec{r}_0}{|\vec{r}_0|^2}+\frac{\Delta\vec{r}\cdot\Delta\vec{r}}{|\vec{r}_0|^2}}}$$

$$\frac{|\Delta\vec{r}|}{|\vec{r}_0|} \ll 1$$

$$\approx -k\left[1+\frac{\Delta\vec{r}\cdot\vec{r}_0}{|\vec{r}_0|^2}+\dots-1\right]\cdot(\vec{r}_0+\Delta\vec{r})\left(1-\frac{\Delta\vec{r}\cdot\vec{r}_0}{|\vec{r}_0|^2}+\dots\right)$$

keep highest order terms

$$\Rightarrow \vec{F} = -k\frac{\Delta\vec{r}\cdot\vec{r}_0}{|\vec{r}_0|^2}\vec{r}_0$$

$$\vec{F} = -k\,\hat{\lambda}_0(\hat{\lambda}_0\cdot\Delta\vec{r})$$

$$[\vec{F}] = -k[\hat{\lambda}_0][\hat{\lambda}_0]^T[\Delta\vec{r}] = -k\left[[\lambda_0][\lambda_0]^T\right][\Delta\vec{r}]$$

<u>Aside</u>: $[\hat{\lambda}_0][\hat{\lambda}_0]^T \underset{2D}{=} \begin{bmatrix}\lambda_x\\ \lambda_y\end{bmatrix}\begin{bmatrix}\lambda_x & \lambda_y\end{bmatrix}$

$\lambda_x = \cos\theta$
$\lambda_y = \sin\theta$

$$= \begin{bmatrix}\lambda_x\lambda_x & \lambda_x\lambda_y\\ \lambda_y\lambda_y & \lambda_y\lambda_y\end{bmatrix}$$

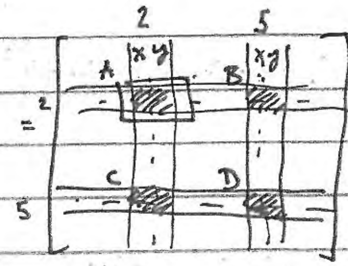
$$m_2\begin{bmatrix}\ddot{x}_2\\ \ddot{y}_2\end{bmatrix} = -k\begin{bmatrix}\lambda_x^2 & \lambda_x\lambda_y\\ \lambda_x\lambda_y & \lambda_y^2\end{bmatrix}\begin{bmatrix}\Delta x_2-\Delta x_5\\ \Delta y_2-\Delta y_5\end{bmatrix} + \text{contribution other springs}$$

$$[\Delta\vec{r}] = \begin{bmatrix}\Delta x_2-\Delta x_5\\ \Delta y_2-\Delta y_5\end{bmatrix}$$

Put together in system eqns

$$M \Delta\ddot{\vec{x}} = -K \Delta\vec{x}$$

$$\begin{bmatrix} m_1 & & & & 0 \\ & m_1 & & & \\ & & m_2 & & \\ & & & m_2 & \\ 0 & & & & \ddots \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \\ \Delta x_2 \\ \Delta y_2 \\ \vdots \end{bmatrix}^{\cdot\cdot} =$$



$$\begin{bmatrix} \Delta x_1 \\ \Delta y_1 \\ \Delta x_2 \\ \Delta y_2 \\ \vdots \end{bmatrix}$$

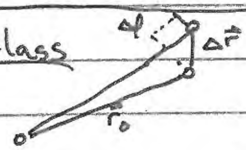
A, D $\begin{bmatrix} \hat{k}_{11} & \hat{k}_{12} \\ \hat{k}_{12} & \hat{k}_{22} \end{bmatrix}$ ← $\hat{k}$ matrix for 2-5 spring ← effect of m_2 on itself

B, C $\begin{bmatrix} -\hat{k}_{11} & -\hat{k}_{12} \\ -\hat{k}_{12} & \hat{k}_{22} \end{bmatrix}$ ← effect of m_5 on m_2

"Assembly" of K

add up K for each spring

Last Class



$$\Delta l \approx \frac{\Delta\vec{r}\cdot\vec{r}_0}{|\vec{r}_0|}$$

$$\vec{F} = -K \frac{\Delta\vec{r}\cdot\vec{r}_0}{|\vec{r}_0|} \frac{\vec{r}_0}{|\vec{r}_0|} = \boxed{\vec{F} = -K\,\hat{\lambda}_0\hat{\lambda}_0\cdot\Delta\vec{r}}$$

Constrained Systems (1 of n) $\boxed{11/4/13}$

Recall: for ~~each~~ each particle

$$\vec{F}_i = m_i \vec{a}_i$$
$$\Rightarrow \vec{a}_i = \vec{F}_i / m_i$$

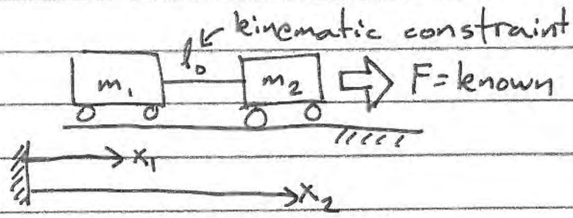
if know $\vec{F}_i = \vec{F}_i(\vec{r}_i, \vec{v}_i, t, \text{parameters})$

known fn.

$\Rightarrow$ can integrate to get soln.

ode45

what if we have kinematic constraints?



FBDs

Problematic:

e.g. $\Sigma F = ma$

~~F=2~~ $F - T = m_1 \ddot{x}_1$

$\ddot{x}_1 = (F - T)/m_1$

① Naive approach: start with LMB & add kinematic constraints

LMB 2: $F - T = m_2 \ddot{x}_2$ ①

LMB 1: $T = m_1 \ddot{x}_1$ ②

Kin const: $x_2 - x_1 = l_0 \Rightarrow \ddot{x}_2 - \ddot{x}_1 = 0$ ③

3 eqns for $\ddot{x}_1, \ddot{x}_2, T$ at every instant in time

$$\left[\begin{array}{cc|c} m_1 & 0 & -1 \\ 0 & m_2 & 1 \\ \hline -1 & 1 & 0 \end{array}\right] \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ T \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}$$

$\Rightarrow$ put inside RHS file for ode45 $\Rightarrow$ numerical solution

accel & T

forces and vel^2 terms (also vel terms)

Shortcuts:

a) Add eqns ① & ②

$\Rightarrow$ eliminates T

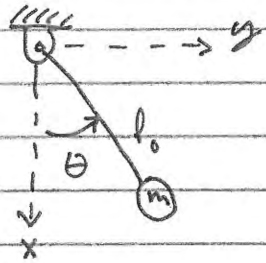
set $\ddot{x} = \ddot{x}_1 = \ddot{x}_2$

b) LMB for syst:

$\Rightarrow F = m_1 \ddot{x}_1 + m_2 \ddot{x}_1$

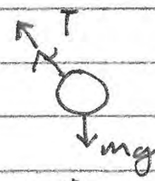
---

ex) Simple Pendulum



$\downarrow g$ whats the motion?

FBD



LMB $\quad \sum \vec{F} = m\vec{a}$

$$mg\,\hat{\imath} + T(-\hat{e}_r) = m\ddot{\vec{r}}$$

$\hookrightarrow \hat{e}_r = \vec{r}/|\vec{r}|$

$\hookrightarrow x\hat{\imath} + y\hat{\jmath}$

$$\ddot{\vec{r}} = g\hat{\imath} - T\,\vec{r}/|\vec{r}|m \quad \star$$

$\hookrightarrow$?

Naive method:

$$\star \Rightarrow \ddot{x} = g - \frac{Tx}{\sqrt{x^2+y^2}\,m} \quad ①$$

$$\ddot{y} = -\frac{Ty}{m\sqrt{x^2+y^2}} \quad ②$$

$$x^2 + y^2 = l_0^2 \Rightarrow 2x\dot{x} + 2y\dot{y} = 0$$

$$\Rightarrow x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 = 0 \quad ③$$

$$\Rightarrow \underbrace{\begin{bmatrix} m & 0 & \frac{x}{\sqrt{x^2+y^2}} \\ 0 & m & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \end{bmatrix}}_{\begin{bmatrix} M & J \\ J' & 0 \end{bmatrix}} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ T \end{bmatrix} = \begin{bmatrix} mg \\ 0 \\ \frac{-\dot{x}^2+\dot{y}^2}{\sqrt{x^2+y^2}} \end{bmatrix}$$

← can solve at every instance inside RHS for ode 45

Shortcuts

1 generalized coord

1 DOF

LMB: $\vec{F} = m\vec{a}$

$-T\hat{e}_r + mg\,\hat{\imath} = m\vec{a}$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$r = \text{const} \Rightarrow \dot{r} = 0,\ \ddot{r} = 0$

$$\{-T\hat{e}_r + mg\,\hat{\imath} = m[r\ddot{\theta}\,\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r]\}$$

$$\{\}\cdot\hat{e}_\theta \Rightarrow \cancel{m}g\underbrace{\hat{\imath}\cdot\hat{e}_\theta}_{-\sin\theta} = \cancel{m}r\ddot{\theta}$$

$$\boxed{\ddot{\theta} + \frac{g}{r}\sin\theta = 0}$$

Method 3

AMB/o

$$\sum \vec{M}_{/o} = \dot{\vec{H}}_{/o}$$

$$\vec{r}_{/o} \times [-T\hat{e}_r + mg\,\hat{\imath}] = \vec{r} \times \overbrace{(r\ddot{\theta}\,\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r)}^{\vec{a}}\, m$$

$\vec{r}\times\hat{e}_r = 0$

$\vec{r}\times\hat{\imath} = -\sin\theta$

$$\{-mg\sin\theta\,\hat{k} = r^2\ddot{\theta}\,m\,\hat{k}\}$$

$$\{\}\cdot\hat{k} \Rightarrow \boxed{\ddot{\theta} + \frac{g}{r}\sin\theta = 0}$$

Method 4 : $E_T = \text{const}$

$$\Rightarrow \dot{E}_T = 0 \Rightarrow \ddot{\theta} + \frac{g}{r}\sin\theta = 0$$

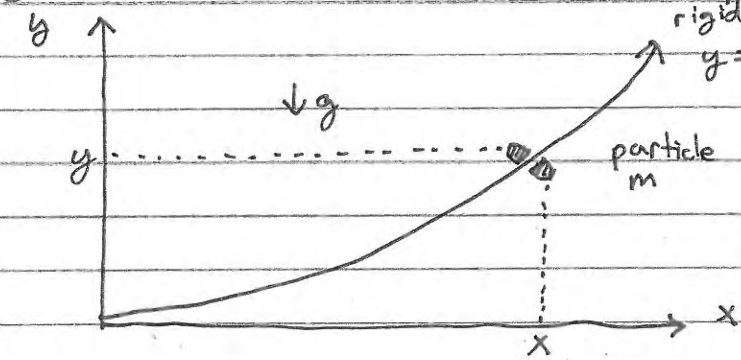
Method 5 | Lag. Eqns.

11/6/13

① Particle Kin. const

② Rigid objects

③ Pendulum

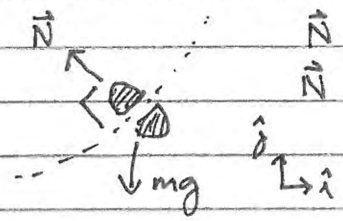


given $x, \dot{x}$

find $\ddot{x}$

EOM

Newton Approach (Newton-Euler Approach)



$\vec{N} \perp$ wire

$\vec{N} \perp \vec{v}$

LMB: $\sum \vec{F} = m\vec{a}$

$$\vec{N} - mg\hat{j} = m\vec{a} \quad ①$$

Generalized coord: x

$$\Rightarrow y = cx^2$$

Kinematics

$$\vec{r} = x\hat{i} + y\hat{j}$$
$$= x\hat{i} + cx^2\hat{j}$$
$$\vec{v} = \dot{\vec{r}} = \dot{x}\hat{i} + 2c\dot{x}x\hat{j}$$
$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \ddot{x}\hat{i} + (2c\dot{x}^2 + 2cx\ddot{x})\hat{j}$$

$\{①\} \cdot \vec{v}$

$\Rightarrow \cancel{\vec{v} \cdot \vec{N}}^{\,0}$

$-mg\hat{j} \cdot \vec{v} = m\vec{a} \cdot \vec{v}$

$$-\cancel{m}g\,2c\cancel{\dot{x}}x = \cancel{m}\left[\cancel{\dot{x}}\ddot{x} + 4c^2\dot{x}^{\cancel{3}\,2}x + 4c^2x^2\cancel{\dot{x}}\ddot{x}\right]$$

$$-g\,2cx = \ddot{x} + 4c^2\dot{x}^2x + 4c^2x^2\ddot{x}$$

$$\Rightarrow \underbrace{\ddot{x} = f(x, \dot{x}, c)}_{\text{E.O.M.}}$$

Method: just like pendulum

$\{\vec{F} = m\vec{a}\} \cdot [\text{vector} \perp \text{to const force}]$

Lag. Eqns

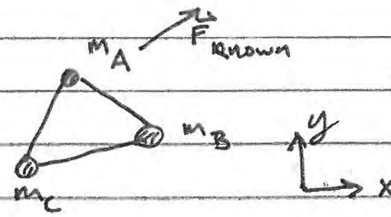
$$\mathcal{L} = E_K - E_P = \text{"}T - V\text{"}$$
$$= \tfrac{1}{2}mv^2 - mgh$$
$$= \tfrac{1}{2}m\left(\dot{x}^2 + (2c\dot{x}x)^2\right) - mgcx^2$$
$$! \quad = \tfrac{1}{2}m(\dot{x}^2 + 4c^2\dot{x}^2x^2) - mgcx^2$$

L.E.

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

$$\left[4mc^2\dot{x}^2x - 2mgcx\right] - \frac{\partial}{\partial t}\left[m\dot{x} + 4c^2\dot{x}x^2m\right] = 0$$

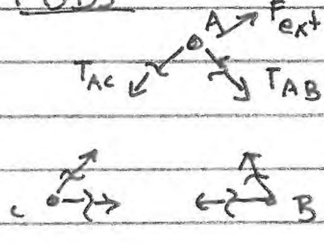
$\Rightarrow \ddot{x} = f(x, \dot{x}, c)$ ✓ same as previous method ex) How does this move?



rigid massless rods

Set up the DAEs (Differential Algebraic Eqs)

FBDs



Kin Constraints

$|\vec{r}_{AB}| = l_{AB} = \text{const}$

$|\vec{r}_{BC}| =$

$|\vec{r}_{CA}| =$

LMB:

$$\left.\begin{array}{l} \sum\limits_{\text{on } A} \vec{F} = m\vec{a}_A \\ \quad\vdots \end{array}\right\}$$

6 scalar eqns for $\ddot{x}_A, \ddot{y}_A$, $\ddot{x}_B, \ddot{y}_B$, $\ddot{x}_C, \ddot{y}_C$

6 eqns $\rightarrow$ 9 unknowns $\Rightarrow$ need 3 more indp eqns (constraints)

Kin Const:

e.g. $\vec{r}_{AB} \cdot \vec{r}_{AB} = \text{const}$

$\Rightarrow \frac{d^2}{dt^2}\left((x_B - x_A)^2 + (y_B - y_A)^2\right) = 0$

same for other bars $= 0$

$\Rightarrow$ 3 more eqns

★★★ Aside: Matthew Cook "Two Neurons can control a bicycle"

$\Rightarrow$ 9 eqns for $\ddot{x}_A, \ddot{y}_A, \ddot{x}_B, \ddot{y}_B, \dots$

T_{AB}, T_{BC}, T_{AC}

in terms of $\vec{F}_{ext}, x_A, y_A, x_B \cdots$

$\dot{x}_A, \dot{y}_A, \dots$

ex) constraint: $x = 7$

$\ddot{x} = 0$

I.C. $x_0 = 7$ & $\ddot{x} = 0 \Rightarrow x = 7$

$\dot{x}_0 = 7$ all t

2D Rigid Objects $\vec{\omega}, \vec{L}, \vec{H}$ & E_K 11/8/13

Consider lots of pt masses w/enough massless bars so the object is rigid

$$\boxed{\# b \geq (\# \text{masses}) * 2 - 3}$$

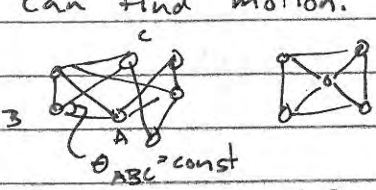
read about trusses in RP (or any statics book)

If too many bars

$\Rightarrow$ incompatability or indeterminary

Still can find motion!

ex)



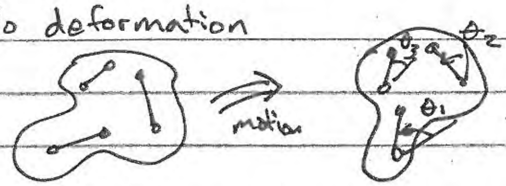
Naive approach: DAEs

<u>Rigid object</u> rigid "body"

all lengths between pts const in time.

all marked angles between line segments const in time

No deformation

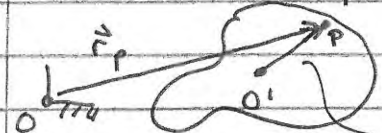


$\theta_1 = \theta_2 = \theta_3 = \theta$

↑ the rotation of the object

$$\boxed{\begin{aligned} \omega &= \text{angular velocity} = \dot\theta \\ \vec\omega &= \omega\hat{k} = \dot\theta\hat{k} \end{aligned}}$$

O' = material pt on object



sometimes O' = * C.O.M. / * hinge

$\vec r_{P/O'} = \vec r_{O'P}$

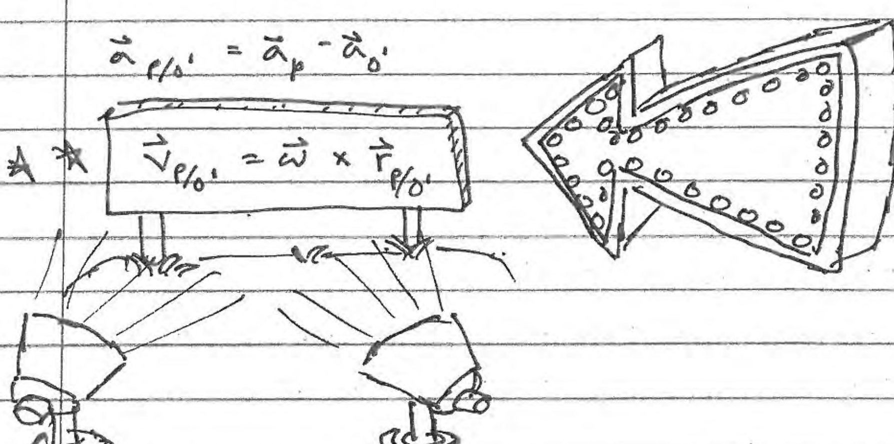
$$\vec r_P = \vec r_{O'} + \vec r_{P/O'}$$

$$\vec v_P = \vec v_{O'} + \overbrace{\frac{\partial}{\partial t}\vec r_{P/O'}}^{\vec v_{P/O'}}$$

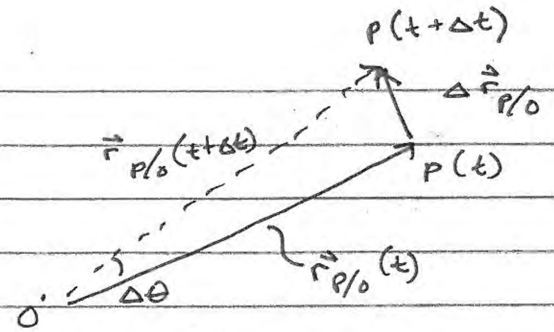
$$\vec a_P = \vec a_{O'} + \underbrace{\frac{\partial}{\partial t}\vec v_{P/O'}}_{\vec a_{P/O'}}$$

$$\vec v_{P/O'} \equiv \vec v_P - \vec v_{O'}$$

$$\vec a_{P/O'} = \vec a_P - \vec a_{O'}$$



$$\vec v_{P/O'} = \vec\omega \times \vec r_{P/O'}$$



p and o' on a rigid object

$$\Delta \vec{r}_{p/o'} = \left|\vec{r}_{p/o'}\right| \cdot \Delta\theta \cdot \underbrace{\hat{n}}_{\perp \text{ to } \vec{r}_{p/o'}}$$

$$= \left[(\Delta\theta)\hat{k}\right] \times \vec{r}_{p/o'}$$

$$\vec{v}_{p/o'} = \frac{\Delta \vec{r}_{p/o'}}{\Delta t} = \frac{\Delta\theta}{\Delta t} \quad \hat{k} \times \vec{r}_{p/o'}$$

$$= \vec{\omega} \times \vec{r}_{p/o'}$$

$$\boxed{\begin{array}{l}\text{RP book}\\ \text{Taylor book}\end{array}}$$

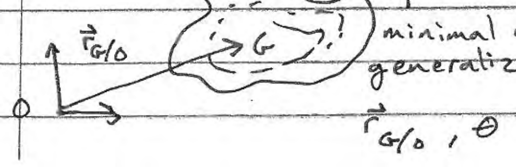
$$\vec{a}_{p/o'} = \frac{\partial}{\partial t} \vec{v}_{p/o'}$$
$$= \frac{\partial}{\partial t}\left(\vec{\omega} \times \vec{r}_{p/o'}\right)$$
$$= \dot{\vec{\omega}} \times \vec{r}_{p/o} + \vec{\omega} \times \left(\vec{v}_{p/o'}\right)$$

$$\vec{a}_{p/o'} = \dot{\vec{\omega}} \times \vec{r}_{p/o'} + \left(\vec{\omega}\right) \times \left(\vec{\omega} \times \vec{r}_{p/o'}\right)$$
$$= \cancel{\ddot{\theta}\,\hat{k} \times \vec{r}_{p/o'} + \dot{\theta}\,\hat{k}}$$

$$\ddot{\theta}\,\hat{k} = \dot{\vec{\omega}} = \vec{\alpha} = \text{ang. accl of object}$$

$$\boxed{\vec{a}_{p/o'} = \ddot{\theta}\,\hat{k} \times \vec{r}_{p/o'} + -\omega^2 \vec{r}_{p/o'}}$$

Look at object & calc motion quantities



minimal coord
generalized coord
$\vec{r}_{G/o}$, θ

$$\vec{L} = \sum m_i \vec{v}_i = m_{tot}\,\vec{v}_G$$

$$\dot{\vec{L}} = m_{tot}\,\vec{a}_G$$

$$\vec{H}_{/c} = \underbrace{\vec{r}_{G/c} \times (m_{tot}\vec{v}_G)}_{\vec{H}_{G/c}} + \underbrace{\sum \vec{r}_{i/G} \times m_i \vec{v}_{i/G}}_{\vec{H}_{/G}}$$

$$\vec{H}_{/G} = \sum \vec{r}_{i/G} \times (m_i \vec{\omega} \times \vec{r}_{i/G}) = \sum r_{i/G}^{\,2}\, \omega m_i \hat{k}$$

$$= \omega \hat{k} \sum m_i r_i^2$$

I^G = moment of inertia about G

$= \sum m_i r_i^2$

$= \int r^2\, dm$ ← continuous system

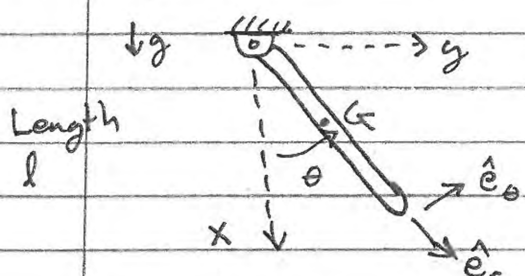
$$\boxed{\vec{H}_{/c} = \vec{r}_{G/c} \times m_{tot} \vec{v}_G + I^G \dot{\theta} \hat{k}}$$

$$\boxed{\dot{\vec{H}}_{/c} = \vec{r}_{G/c} \times m_{tot} \vec{a}_G + I^G \ddot{\theta} \hat{k}}$$

$$\boxed{E_K = \frac{1}{2} m_{tot} v_G^2 + \frac{1}{2} I^G \dot{\theta}^2}$$

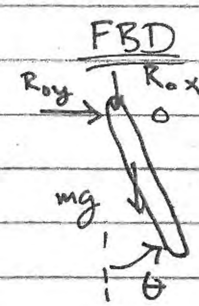
⇒ can find EOM for rigid objects

eg) Pendulum



$I^G = \frac{1}{12} m l^2$

FBD



Angular Mom Balance /o

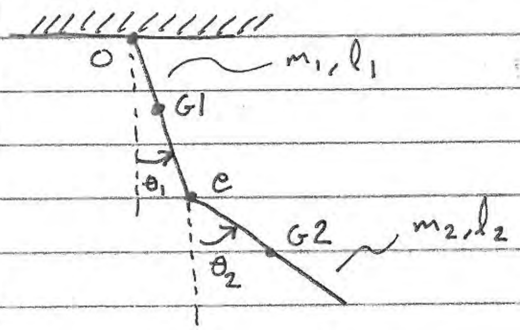
$$\sum \vec{M}_{/0} = \dot{\vec{H}}_{/0}$$

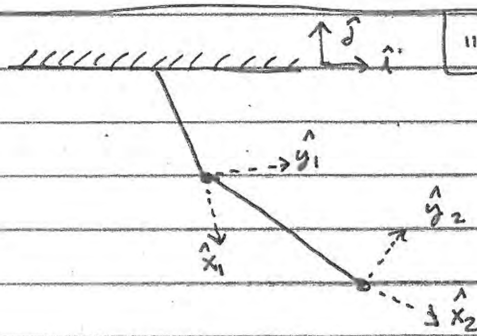
$$\vec{r}_{G/0} \times (mg\,\hat{i}) = \vec{r}_{G/0} \times (m\vec{a}_G) + I^G \ddot{\theta}\,\hat{k}$$

$\llcorner \frac{l}{2}\hat{e}_r$

$$\llcorner\!\!\to \vec{a}_G = \ddot{\theta}\,\hat{k} \times \vec{r}_{G/0} - \dot{\theta}^2 \vec{r}_{G/0}$$

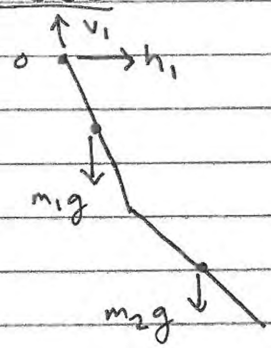
$$\Rightarrow \boxed{mg\frac{l}{2}\sin\theta = -\underbrace{\left(I^G + m\frac{l^2}{4}\right)}_{I^0}\ddot{\theta}}$$





11/11/13

FBD1

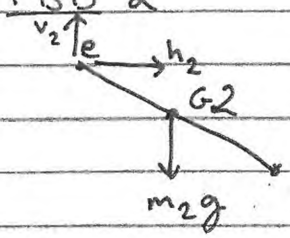


$$\sum \vec{M}_{/0} = \dot{\vec{H}}_{/0}$$

$$\sum \vec{M}_{/0} = \vec{r}_{G1/0} \times (-m_1 g\,\hat{j}) + \vec{r}_{G2/0} \times (-m_2 g\,\hat{j})$$

$$\dot{\vec{H}}_{/0} = \vec{r}_{G1/0} \times (m_1 \ddot{\vec{r}}_{G1/0}) + I_{1/G1}\ddot{\theta}_1\,\hat{k}$$
$$+ \vec{r}_{G2/0} \times (m_2 \ddot{\vec{r}}_{G2/0}) + I_{2/G2}\ddot{\theta}_2\,\hat{k}$$

FBD 2



$$\sum \vec{M}_{/e} = \dot{\vec{H}}_{/e}$$

$$\sum \vec{M}_{/e} = \vec{r}_{G2/e} \times \left(-m_2 g\, \hat{j}\right)$$

$$\dot{\vec{H}}_{/e} = \vec{r}_{G/2/e} \times \left(m_2 \ddot{\vec{r}}_{G2/e}\right) + I_{2/G2}\, \ddot{\theta}_2\, \hat{k}$$

$$\hat{x}_1 = \sin\theta_1\, \hat{\imath} - \cos\theta_1\, \hat{\jmath}$$
$$\hat{y}_1 = \cos\theta_1\, \hat{\imath} + \sin\theta_1\, \hat{\jmath}$$

$$\hat{x}_2 = \sin\theta_2\, \hat{\imath} - \cos\theta_2\, \hat{\jmath}$$
$$\hat{y}_2 = \cos\theta_2\, \hat{\imath} + \sin\theta_2\, \hat{\jmath}$$

$$\dot{\hat{x}}_1 = \frac{\partial}{\partial t}\left(\hat{x}_1\right) = \frac{\partial}{\partial t}\left(\sin\theta_1\, \hat{\imath} - \cos\theta_1\, \hat{\jmath}\right)$$
$$\dot{\hat{x}}_1 = \dot{\theta}_1 \cos\theta_1\, \hat{\imath} + \dot{\theta}_1 \sin\theta_1\, \hat{\jmath} = \dot{\theta}_1\, \hat{y}_1$$

ASIDE $\dot{q} = \vec{\omega} \times q$

$$\dot{\hat{x}}_1 = \left(\dot{\theta}\, \hat{k}\right) \times \left(\hat{x}_1\right) = \dot{\theta}\, \hat{y}_1$$

$$\begin{cases} \dot{\hat{x}}_1 = \dot{\theta}_1\, \hat{y}_1 \\ \dot{\hat{y}}_1 = -\dot{\theta}_1\, \hat{x}_1 \end{cases} \qquad \begin{cases} \dot{\hat{x}}_2 = \dot{\theta}_2\, \hat{y}_2 \\ \dot{\hat{y}}_2 = -\dot{\theta}_2\, \hat{x}_2 \end{cases}$$

$$\begin{cases} \ddot{\hat{x}}_1 = \ddot{\theta}_1\, \hat{y}_1 + \dot{\theta}_1\, \dot{\hat{y}}_1 = \ddot{\theta}_1\, \hat{y}_1 - \dot{\theta}_1^2\, \hat{x}_1 \\ \ddot{\hat{y}}_1 = \cancel{-\ddot{\theta}_1 \hat{y}_1 - \dot{\theta}_1^2\, \hat{y}_1} = -\ddot{\theta}_1\, \hat{x}_1 - \dot{\theta}_1^2\, \hat{y}_1 \end{cases}$$

$$\ddot{\hat{x}}_2 = \ddot{\theta}_2\, \hat{y}_2 + -\dot{\theta}_2^2\, \hat{x}_2$$
$$\ddot{\hat{y}}_2 = -\ddot{\theta}_2\, \hat{x}_2 - \dot{\theta}_2^2\, \hat{y}_2$$

$\vec{r}_{G1/O} = \frac{1}{2} l_1 \hat{x}_1$ $\qquad$ $\ddot{\vec{r}}_{G1/O} = \frac{1}{2} l_1 \ddot{\hat{x}}$

$\vec{r}_{G2/O} = l_1 \hat{x}_1 + \frac{1}{2} l_2 \hat{x}_2$ $\qquad$ $\ddot{\vec{r}}_{G2/O} = l_1 \ddot{\hat{x}}_1 + \frac{1}{2} l_2 \ddot{\hat{x}}_2$

$\vec{r}_{G2/e} = \frac{1}{2} l_2 \hat{x}_2$ $\qquad$ $\vec{r}_{G2/e} = \frac{1}{2} l_2 \ddot{\hat{x}}_2$

$$\begin{aligned}
\Sigma \vec{M}_{/O} &= -m_1 g \left(\vec{r}_{G1/O} \times \hat{j}\right) - m_2 g \left(\vec{r}_{G2/O} \times \hat{j}\right) \\
&= -m_1 g \left(\frac{l_1}{2} \sin \theta_1\right) - m_2 g \left(\left(l_1 \hat{x}_1 + \frac{1}{2} l_2 \hat{x}_2\right) \times \hat{j}\right) \\
&= -m_1 g \left(\frac{l_1}{2} \sin \theta_1\right) - m_2 g \left(l_1 \sin \theta + \frac{1}{2} l_2 \sin \theta_2\right)
\end{aligned}$$

$I_{1/G1} = \frac{1}{12} m_1 l_1^2$ $\qquad$ $I_{2/G2} = \frac{1}{12} m_2 l_2^3$

★ Double angle problem Angular momentum balance
Bhounsule,

★ **PLOT TWIST : Controls !** ★ ★ ★ ★ 11/11/13

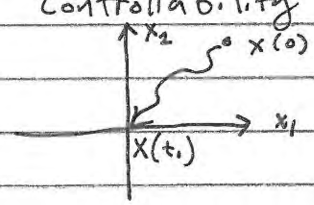
Exam Thursday @ 7:30 - 9:30
Statler Hall Rm 265

Topics: root locus, discrete-time, Bode plot, gain/phase margin, lead/lag design, Nyquist stability

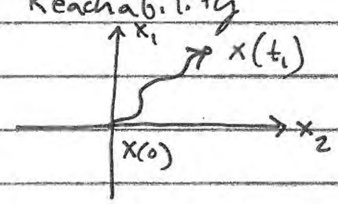
Review: ~~Th~~ Tuesday 12:50 – ? Thurston 202

Today

Controllability



Reachability



2. Overview

a) <u>Motivation</u>: manipulate system to exhibit behavior

b) <u>Goals</u>: determine whether input u that drives the system from $x(t_0)$ to desired $x(t_1)$

c) <u>Reachability</u>

i) <u>def'n</u> x_f is reachable if there exist u that transfers state $x(t_0) = 0$ to $x(t_1) = x_f$

ii) <u>continuous-time check</u>

x_f is reachable if and only if $x_f \in \operatorname{Im} \mathcal{C}$

where $\mathcal{C} = \begin{bmatrix} B & A^1B & \cdots & A^{n-1}B \end{bmatrix}$

$\dot{x} = Ax + Bu \,,\quad x \in \mathbb{R}^n$

iii) <u>Discrete-time</u> has the same result

for $x(t+1) = Ax(t) + Bu(t)$

d) <u>Controlability</u>

there exists

i) <u>def'n</u> x_0 is controllable if $\exists$ input u that transfers the state from $x(t_0) = x_0$ to $x(t_1) = 0$

ii) <u>continuous-time check</u> same as the reachability check (replace x_f with x_0)

iii) <u>Discrete-time</u> all reachable states are controllable, but not vice versa

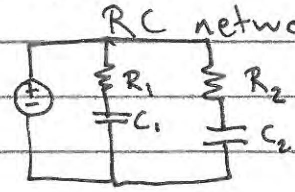
e.g. x_0 controllable $\not\Rightarrow$ x_f reachable

<u>why?</u> A may not be full-rank

3. Discrete-Time Reachability examples

a) overview

<u>obj</u>: find which states x_f are reachable for RC network



continuous time:

$$\dot{x} = \begin{bmatrix} -1/R_1C_1 & 0 \\ 0 & -1/R_2C_2 \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \end{bmatrix} u$$

where x_1, x_2 are voltages of capacitors

Discretized dynamics

$$x(t+1) = \bar{A}\, x(t) + \bar{B}\, u(t)$$

$$\bar{A} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \qquad a_1 = e^{\left(\frac{-1}{R_1C_1}\right)T} \qquad a_2 = e^{\left(\frac{-1}{R_2C_2}\right)T}$$

$$\bar{B} = A^{-1}\left(1 - e^{AT}\right)B$$

$$\bar{B} = \begin{bmatrix} -R_1C_1 & 0 \\ 0 & -R_2C_2 \end{bmatrix} \begin{bmatrix} 1-a_1 & 0 \\ 0 & 1-a_2 \end{bmatrix} B$$

$\bar{B} =$ "MAGIC!"

$$\bar{B} = \begin{bmatrix} -1+a_1 \\ -1+a_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- Determine $x(2)$ in terms of $x(0)$ and u
- Apply the state space eqns

$$x(1) = \bar{A}\, x(0) + \bar{B} u(0)$$
$$x(2) = \bar{A}\, x(1) + \bar{B} u(1)$$

Eliminate $x(1)$

$$x(2) = \bar{A}\left(\bar{A}\, x(0) + \bar{B} u(0)\right) + \bar{B} u(1)$$
$$x(2) = \bar{A}^2 x(0) + \bar{A}\bar{B}\, u(0) + \bar{B}\, u(1)$$

Reachability:

Given $x(0) = 0$, find $u = \{u(0), u(1)\}$

such that $x(2) = x_f$

$$\Rightarrow x(2) = \begin{bmatrix} \bar{A}\bar{B} & \cancel{\bar{A}}\bar{B} \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

if $\mathcal{C} = [\bar{B} \quad \bar{A}\bar{B}]$ is full rank then there exists such a u

$\Rightarrow u = \mathcal{C}^{-1} x_f$

<u>remark</u> won't necessarily stay @ x_f if x_f not an eq. point

ex) $x(3) = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \bar{A}^2\bar{B} \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}$

Does there exist $u = \{u(0), u(1), u(2)\}$ such that $x(3) = x_f$?

<u>YES</u> $\begin{bmatrix} u(2) \\ u(1) \end{bmatrix} = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} \end{bmatrix}^{-1} x_f$

<u>Why</u> $\mathcal{C} = [\bar{B} \quad \bar{A}\bar{B} \quad \dots \quad \bar{A}^{n-1}\bar{B}]$

Cayley - Hamilton Theorm

<u>what</u> if we can't reach x_f in n steps then it's not " reachable

---

When is $\mathcal{C} = [\bar{B} \quad \bar{A}\bar{B}]$ full rank ?

$= \begin{bmatrix} b_1 & a_1 b_1 \\ b_2 & a_2 b_2 \end{bmatrix}$

- linearly independent columns
- $\det \mathcal{C} = b_1 b_2 a_2 - b_1 b_2 a_1$
  $= b_1 b_2 (a_2 - a_1)$

$\Rightarrow$ full rank $a_2 \neq a_1$, $b_1 \neq 0$, $b_2 \neq 0$

What if $a_1 = a_2$

$\Rightarrow \mathcal{C}$ is not full rank

x_f reachable if $x_f \in \operatorname{Im} \mathcal{C} = \operatorname{Im} \begin{bmatrix} b_1 & a_1 b_1 \\ b_2 & a_1 b_2 \end{bmatrix}$

$x_f \in \operatorname{Im} \mathcal{C}$ if $x_f = \alpha \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ where $\alpha \in \mathbb{R}$

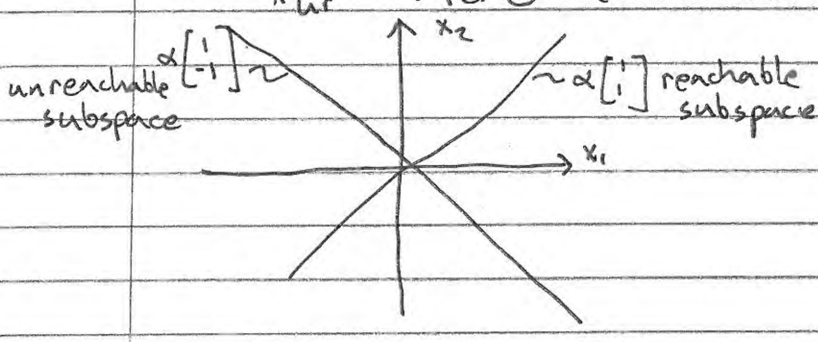
T~~ransfer function~~

⇒ unreachable subspace

x_{ur} s.t. $x_{ur} \perp \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x_{ur} = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$x_{ur} \in \operatorname{Ker} C^T$ (unreachable subspace)



Transfer Function

$$y(t) = x_1(t) + x_2(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

take Z transform

$$\frac{Y(z)}{U(z)} = \frac{b_1}{z-a_1} + \frac{b_2}{z-a_2} = \frac{(b_1+b_2)z - (b_1 a_2 + b_2 a_1)}{(z-a_1)(z-a_2)}$$

⇒ pole @ $z = a_1, a_2$

zero @ $z = \dfrac{b_1 a_2 + b_2 a_1}{b_1 + b_2}$

if $a_1 = a_2$, then zero @ $z = a_1 = a_2$

⇒ pole-zero cancellation

Take-away: not controllable, then pole-zero cancellation

11/13/13 Kinematic Synthesis - LIPSON

- homogenization algorithm - static design
- kinematic synthesis - poorly understood
  analytic methods for special cases
  ⇒ Chebychev
- Robert Willis - The Principles of Mechanisms, 1841
- Kempe How to draw a straight line 1877

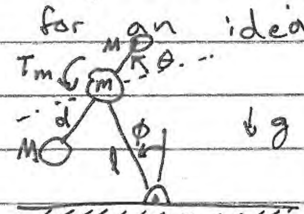
Relaxation Based Simulation of Kinematic Mechanisms

treat each member like spring
⇒ calc force at each node and move "small" amount in that direction
⇒ repeat using "stretch of spring" to calc new forces and new node displacement
⇒ repeat ad infinitum

Can teach analysis, but not synthesis

11/15/13

PSIAKI: Apply Lagrange's Equations w/generalized forces to develop a model for an idealized tight-rope walker



T_m: torque applied by the tight rope walker to the balance beam

generalized coordinates ϕ: tight rope walker lean angle relative to vertical

θ: rotation angle of the balance beam relative to the walkers' hips

$L = T - V$

V → potential energy

T → kinetic energy

Lagrange's eqns (if $T_m = 0$)

$$\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\phi}}\right] - \frac{\partial L}{\partial \phi} = 0 \qquad \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\theta}}\right] - \frac{\partial L}{\partial \theta} = 0$$

If $T_m \neq 0$, a non-conservative force

$$\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\phi}}\right] - \frac{\partial L}{\partial \phi} = Q_\phi$$

$$\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\theta}}\right] - \frac{\partial L}{\partial \theta} = Q_\theta$$

$\begin{bmatrix} Q_\phi \\ Q_\theta \end{bmatrix}$ Generalized force vector corresponding T_m in the generalized coordinate space $\begin{bmatrix} \phi \\ \theta \end{bmatrix}$

Determine Q_ϕ determining the virtual work done by a virtual displacement $\delta\phi$, while holding θ const

By definition $\delta W_\phi = \dot{Q}_\phi \delta\phi = 0$ in this case

$$\Rightarrow Q_\phi = \frac{0}{\delta\phi} = 0$$

Determine Q_θ using virtual work

$$\delta W_\theta = Q_\theta \delta\theta = T_m \delta_\theta$$

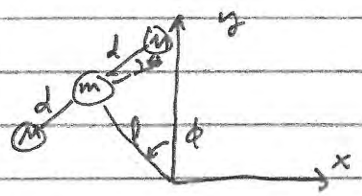
so $Q_\theta = T_m$

Now determine $T + V$ in terms of $\phi, \dot\phi, \theta, \dot\theta$

Get cartesian position coordinates of each mass as functions of d, θ

Differentiate to get their velocities, then $T = \frac{1}{2} m v_{el}^2$

$V = mgh$



Central Mass

$$x_m = -l \sin\phi \qquad V_a = mg\, y_m = mg\, l\cos\phi$$

$$y_m = l\cos\phi$$

$$\dot x_m = -l\dot\phi\cos\phi \qquad T_a = \frac{1}{2} m\left(\dot x_m^2 + \dot y_m^2\right)$$

$$\dot y_m = -l\dot\phi\sin\theta \qquad = \frac{1}{2} m l^2 \dot\phi^2$$

Right mass M

$$x_{MR} = -l\sin\phi + d\cos(\phi + \theta)$$

$$y_{MR} = l\cos\phi + d\sin(\phi+\theta)$$

$$\dot x_{MR} = -l\dot\phi\cos\phi - d(\dot\phi + \dot\theta)\sin(\phi+\theta)$$

$$\dot y_{MR} = -l\dot\phi\sin\phi + d(\dot\phi + \dot\theta)\cos(\phi+\theta)$$

$$V_B = M g\, y_{MR} = M g\left[l\cos\phi + d\sin(\phi+\theta)\right]$$

$$T_B = \frac{1}{2} M\left(\dot x_{MR}^2 + \dot y_{MR}^2\right)$$
$$= \frac{1}{2} M\left[l^2\dot\phi^2 + l d\,\dot\phi(\dot\phi+\dot\theta)\sin\theta + d^2(\dot\phi+\dot\theta)^2\right]$$

flip d

$$\begin{cases} V_c = M g\left[l\cos\phi - d\sin(\phi+\theta)\right] \\ T_c = \frac{1}{2} M\left(l^2\dot\phi^2 - l d\,\dot\phi(\dot\phi+\dot\theta)\sin\theta + d^2(\dot\phi+\dot\theta)^2\right] \end{cases}$$

Totalling T & V

$$T = T_a + T_b + T_c = \frac{1}{2}(m + 2M)\, l^2 \dot{\phi}^2 + M\, d^2(\dot{\phi} + \dot{\theta})^2$$

$$V = V_a + V_b + V_c = (m + 2M)\, l\, g \cos\phi$$

Apply Lagrange

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) \underbrace{- \frac{\partial T}{\partial \phi} + \frac{\partial V}{\partial \phi}}_{-\frac{\partial L}{\partial \phi}} = 0$$

$$\frac{d}{dt}\left[(m+2M)\, l^2 \dot{\phi} + 2M d^2(\dot{\phi} + \dot{\theta})\right] - \left[m + 2M\right] l g \cos\phi = 0$$

$$\star \quad \boxed{\left[(m+2M)\, l^2 + 2M d^2\right] \ddot{\phi} + \left[2M d^2\right] \ddot{\theta} - \left[m + 2M\right] l g \sin\phi = 0}$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}}\right] - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = T_m$$

$$\frac{d}{dt}\left[2M\, d^2(\dot{\phi} + \dot{\theta})\right] = T_m$$

$$\boxed{\left[2M d^2\right] \ddot{\phi} + \left[2M d^2\right] \ddot{\theta} = T_m}$$

Solving for $\ddot{\phi}$ an $\ddot{\theta}$

$$\ddot{\phi} = \frac{g}{l} \sin\phi - \left[\frac{1}{(m+2M)\, l^2}\right] T_m$$

$$\ddot{\theta} = \frac{-g}{l} \sin\phi + \left[\frac{m l^2 + 2M(l^2 + d^2)}{2M d^2 (m+2M)\, l^2}\right] T_m$$

↳ where $T_m = 0$, simplifies to basic pendulum problem

Question

$$\dot{E} \overset{?}{=} T_m \dot{\theta} \quad ?$$

$$\dot{E} = \frac{\partial T}{\partial \dot{\phi}} \ddot{\phi} + \frac{\partial T}{\partial \dot{\theta}} \ddot{\theta} + \frac{\partial V}{\partial \phi} \dot{\phi}$$

$$= \frac{\partial T}{\partial \dot{\phi}} \left[\frac{g}{l} \sin\phi - (\dots) T_m \right] + \frac{\partial T}{\partial \dot{\theta}} \left[\frac{-g}{l} \sin\phi + (\dots) T_m \right) + \frac{\partial V}{\partial \phi} \dot{\phi} \dots = T_m \dot{\theta}$$

where $T_m = 0$, $\dot{E} = 0$.

Grab the balance beam

$\theta = 0$, $\dot{\theta} = 0$, $\ddot{\theta} = 0$

$0 = \frac{-g}{l} \sin\theta + [\dots] T_m$

and solve for T_m and put in $\ddot{\phi}$ eq

$$\ddot{\phi} = \left[\frac{(m+2M) l^2}{(m+2M) l^2 + 2Md^2} \right] \frac{g}{l} \sin\phi$$

larger d, smaller $\ddot{\phi}$

slow down the fall

11/18/13 TODAY

① 2D Rigid Object Dyn. Review

② Double Pendulum ($3/n$, $n = 3?\ 4?$)

(on computer)

2D Rigid Object Summary

① Draw FBDs of parts and collections of parts.

② Pick coords. ($q_i = \theta, x, y$ of parts, usually)

minimal or generalized coord + others

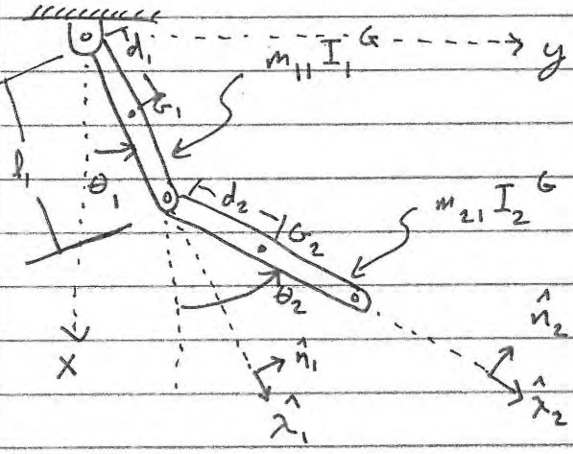
③ Write LMB & AMB for each FBD (3 scalar eqns)

④ Kinematic contraint eqns

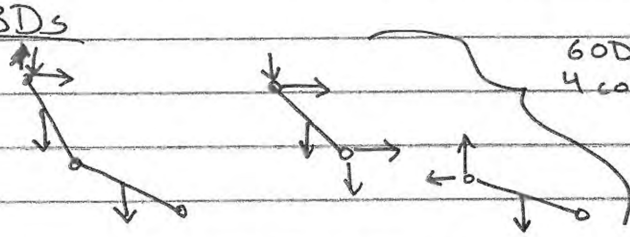
⑤ Given $q_i, \dot{q}_i$, parameters, we $\underbrace{\text{solve for } \ddot{q}_i}_{\text{Eqs of motion (EOM)}}$, and contsraint forces

[Eqs are linear in $\ddot{q}_i$, F_j^{cons}, applied forces]
[non-linear in $\dot{q}_i, q_i$]

Double Pendulum (see also lectures from Nov 11 & 15)



FBDs



6 ODEs
4 constraint eqns

solve for $\ddot{\theta}_1, \ddot{\theta}_2, \ddot{x}_1, \ddot{y}_1, \ddot{x}_2, \ddot{y}_2$, 4 constraint forces [10 things, 10 eqns]

$\boxed{11/20/13}$

MATLAB Code Testing

Checks:

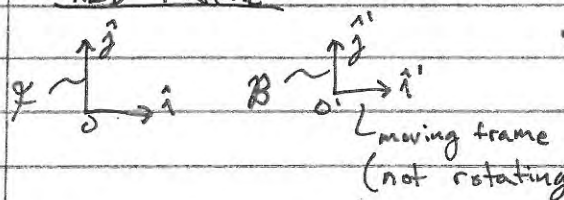
- energy ~~cons~~ conserved ?
- $g=0$, is Ang Mom conserved
- ~~sped~~ special case and extreme cases
- animation

11/22/13 TODAY

① Accel ref frames (intro)

② Pendulum w/ moving base

FIXED FRAME



P = pt. of interest

All of mechanics based on

① $\vec{F} = m\vec{a}$ — calculated in Newtonian frame

② Action and Reaction

③ Internal forces have no net force, no net moment

Consider moving frame

$$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}$$

$$\text{"}\vec{r}\text{"} = \left(x_{O'/O}\,\hat{i} + y_{O'/O}\,\hat{j}\right) + x'_{P/O'}\,\hat{i}' + y'_{P/O'}\,\hat{j}'$$

For non-rotating frame $\hat{i}' = \hat{i}$, $\hat{j}' = \hat{j}$

$$\vec{v} = \dot{\vec{r}}$$

$$\vec{a} = \dot{\vec{v}}$$

$$\boxed{\vec{a} = \vec{a}_{O'/O} + \vec{a}_{P/\mathcal{B}}}$$

$$= \ddot{x}_{P/O'}\,\hat{i}' + \ddot{y}_{P/O'}\,\hat{j}'$$

$\vec{a}_{P/\mathcal{B}}$ = acc. of P w.r.t ref frame $\mathcal{B}$

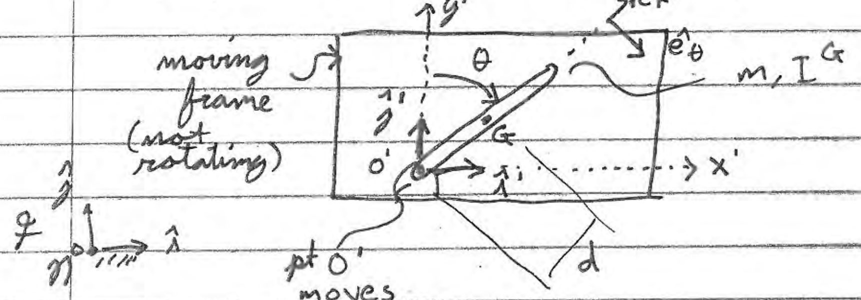
= keep base vectors fixed, change comps.

This is a test of writing in cursive. My name is Bryan Pele.

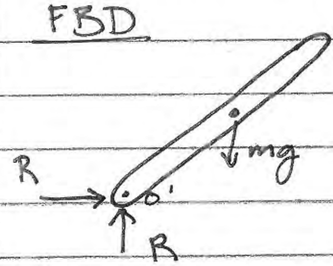
$\vec{a}_{/O'}$ = loose language for

$\vec{a}_{/\beta_r}$ ref. frame that has O' fixed in it, & doesn't rotate

ex) pendulum in moving frame



FBD



AMB $/O'$

$$\sum \vec{M}_{/O'} = \dot{\vec{H}}_{/O'}$$

$$\vec{r}_{G/O'} \times (-mg\hat{j}) = \vec{r}_{G/O'} \times m\vec{a}_G + I^G \ddot{\theta}\hat{k}$$

$\vec{r}_{G/O'}$: $d\hat{e}_r$

$\vec{a}_G$: $\vec{a}_{O'} + \vec{a}_{P/\beta}$

$\vec{a}_{O'}$: $\ddot{x}'\hat{i}' + \ddot{y}'\hat{j}'$

$\vec{a}_{P/\beta}$: $= d\ddot{\theta}\hat{e}_\theta - d\dot{\theta}^2\hat{e}_r$

$$-d\sin(\theta)mg\hat{k} = \vec{r}_{G/O'} \times (m\vec{a}_{O'}) + d^2 m \ddot{\theta}\hat{k} - I^G\ddot{\theta}\hat{k}$$

$$dmg\sin\theta\,\hat{k} = (I + d^2 m)\ddot{\theta}\hat{k} - \vec{r}_{G/O'} \times m\vec{a}_{O'}$$

Special Cases:

1) horiz accel:

$\Rightarrow$ pend balancing

11/25/13

TODAY
① Slight Recap
② Chaplygih Sleigh (Chaplygih)
[③ Rotating coords]

$\swarrow \vec{a}_\beta = \vec{a}_{\beta/\mathcal{F}}$

Recap Frame β accelerates but does not rotate

$$\sum \vec{M}_{/c} = \sum \vec{r}_{i/c} \times m_i \vec{a}_i$$

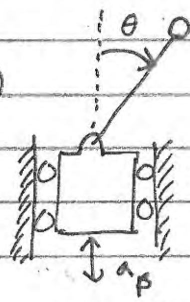
$$\sum \vec{M}^{other}_{/c} + \sum \vec{r}_{i/c} \times m_i \vec{g} = \sum \vec{r}_{i/c} \times m\left[\vec{a}_{o'} + \vec{a}_{i/\beta}\right]$$

$$\sum \vec{M}^{other}_{/c} + \sum \vec{r}_{i/c} \times m\underbrace{\left(\vec{g} + -\vec{a}_{\beta/\mathcal{F}}\right)}_{\text{altered gravity field}} = \sum \vec{r}_{i/c} \times m\vec{a}_{i/\beta}$$

$\hookrightarrow \vec{a}_\beta$

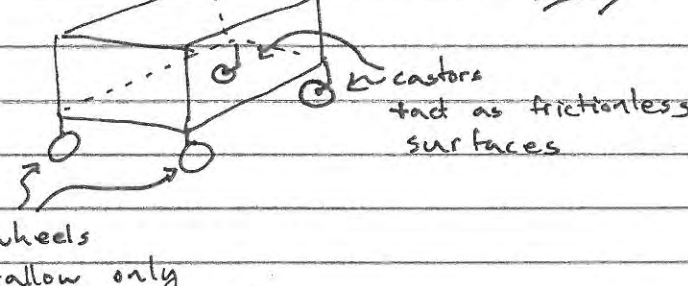
$\uparrow$ if we use accel rel to moving frame

ex)

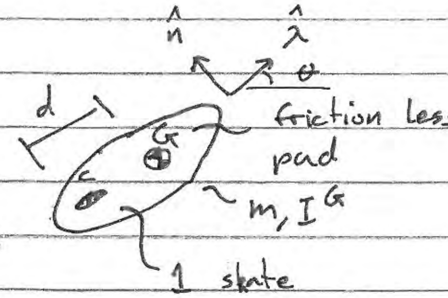


$$\Rightarrow \ddot{\theta} + \overbrace{\frac{g+a}{l}}^{\text{"altered gravity}} \sin\theta = 0$$

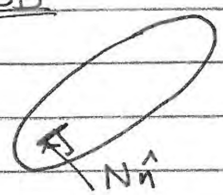
Chaplygih Sleigh



$\Longrightarrow$



FBD



"minimal coord" : $\theta,\ v_c$

$\vec{r}_G = \vec{r}_c + d\hat{\lambda}$

$\vec{v}_G = \vec{v}_c + \vec{v}_{G/c}$

$\vec{a}_G = \vec{a}_c + \vec{a}_{G/c}$

$\dot{\vec{r}}_c = \dot{x}_c\hat{\imath} + \dot{y}_c\hat{\jmath} = v_c\hat{\lambda}$

$\quad \hookrightarrow \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$

ASIDE:
$\vec{A}$ such that $|\vec{A}|$ const
$\dot{\vec{A}} = \dot{\theta}\hat{k} \times \vec{A}$

$\vec{a}_G = \vec{a}_c + \vec{a}_{G/c}$

$\quad \llcorner\ -d\omega^2\hat{\lambda} + d\dot{\omega}\hat{n}$

$\llcorner\ \vec{a}_c = \frac{d}{dt}\vec{v}_c = \frac{d}{dt} v_c\hat{\lambda}$

$\quad = \dot{v}_c\hat{\lambda} + v_c\dot{\hat{\lambda}}$

$\qquad \llcorner\ \vec{\omega} \times \hat{\lambda} = \dot{\theta}\hat{k} \times \hat{\lambda} = \dot{\theta}\hat{n}$

LMB $\cdot\,\hat{\lambda}$ and AMB/c

LMB $\cdot\,\hat{\lambda}$

$\{\Sigma\vec{F} = m\vec{a}_G\}\cdot\hat{\lambda}$

$0 = \left[\left(\dot{v}_c\hat{\lambda} + \cancel{v_c\dot{\theta}\hat{n}}\right) + \left(-d\omega^2\hat{\lambda} + \cancel{d\dot{\omega}\hat{n}}\right)\right]\cdot\hat{\lambda}$

$$\boxed{\dot{v}_c = d\omega^2}$$

AMB/c

$\Sigma M_{/c} = \dot{\vec{H}}_{/c}$

$0 = \vec{r}_{G/c} \times m\vec{a}_G + I^G\dot{\omega}\hat{k}$

$0 = (d\hat{\lambda}) \times m\left(\dot{v}_c\hat{\lambda} + v_c\dot{\theta}\hat{n} - d\omega^2\hat{\lambda} + d\dot{\omega}\hat{n}\right) + I^G\ddot{\theta}\hat{k}$

$\{0 = dm\dot{\theta}v_c\hat{k} + d^2m\dot{\omega}\hat{k} + I^G\ddot{\theta}\hat{k}\}\cdot\hat{k}$

$0 = md\dot{\theta}v_c + d^2m\dot{\omega} + I^G\ddot{\theta}$

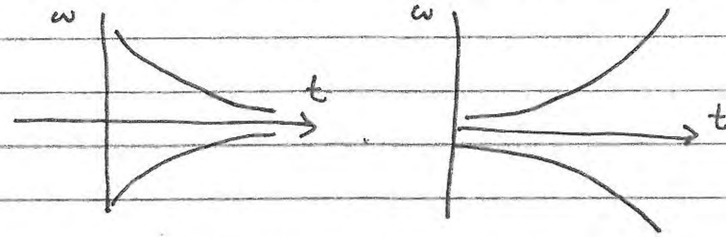
$0 = md\dot{\theta}v_c + (d^2m + I^G)\ddot{\theta} \Rightarrow \boxed{\ddot{\theta} = \frac{-dmv_c\dot{\theta}}{d^2m + I^G}}$

EOM $\quad \dot{v}_c = d\omega^2$ ①

$$\dot{\omega} = \frac{-d m v_c \omega}{d^2 m + I^G} \quad ②$$

$\dot{x}_c = v_c \cos\theta$ ③ $\qquad \dot{y}_c = v_c \sin\theta$ ④

Overall behavior iff $v_c > 0$ $\qquad\qquad$ $v_c < 0$



Nonholonomic : ice skates, rolling

configuration variables > number of ways it can move

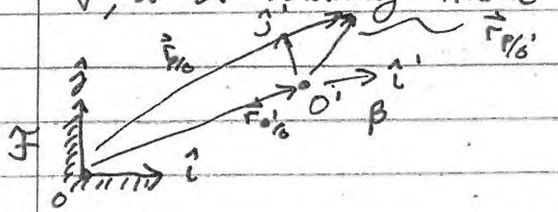
eg) need x, y, θ to define position $\qquad$ can only move tangential or rotate about one point

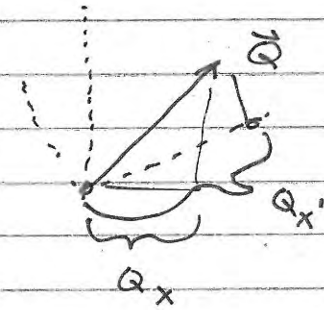
derivation of Lagrange does not work because non integrable

NOTE: Thanksgiving Wednesday

$\vec{v}$, $\vec{a}$ & rotating frame

11/27/13





$$\vec{Q} = \vec{Q}$$

$$\boxed{\vec{Q}_x \hat{\imath} + Q_y \hat{\jmath} = Q_{x'} \hat{\imath} + Q_{y'} \hat{\jmath}'}$$

$\vec{Q}$ can be represented in any reference frame

Time derivative

$$\dot{\vec{Q}} \equiv {}^{\mathcal{F}}\dot{\vec{Q}} \equiv \dot{Q}_x \hat{\imath} + \dot{Q}_y \hat{\jmath}$$

Derivative with respect to a moving frame

$${}^{\beta}\dot{\vec{Q}} = \dot{Q}_{x'} \hat{\imath}' + \dot{Q}_{y'} \hat{\jmath}' = \underbrace{\dot{Q}_{x''} \hat{\imath}'' + \dot{Q}_{y''} \hat{\jmath}''}_{\substack{\text{a different coordinate system} \\ \beta'' \text{ "glued" to } \beta}}$$

Frame vs. Coordinate system

↳ set of all coordinate systems that move together

Relationship between ${}^{\mathcal{F}}\dot{\vec{Q}}$ and ${}^{\beta}\dot{\vec{Q}}$

$${}^{\mathcal{F}}\dot{\vec{Q}} = \dot{Q}_x \hat{\imath} + \dot{Q}_y \hat{\jmath} = \frac{d}{dt}\left(Q_{x'} \hat{\imath}' + Q_{y'} \hat{\jmath}'\right)$$

$$= \dot{Q}_{x'} \hat{\imath}' + Q_{x'} \dot{\hat{\imath}}' + \dot{Q}_{y'} \hat{\jmath}' + Q_{y'} \dot{\hat{\jmath}}'$$

$\vec{\omega} \equiv \vec{\omega}_{\beta/\mathcal{F}}$

$\dot{\hat{\imath}}' = \vec{\omega} \times \hat{\imath}'$

$\dot{\hat{\jmath}}' = \vec{\omega} \times \hat{\jmath}'$

ASIDE: $\hat{\imath}' = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$

$\dot{\hat{\imath}}' = -\dot{\theta}\sin\theta\,\hat{\imath} + \dot{\theta}\cos\hat{\jmath}$

$= \dot{\theta}(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$

$= \dot{\theta}\,\hat{k} \times \hat{\imath}'$

$$\boxed{{}^{\mathcal{F}}\dot{\vec{Q}} = {}^{\beta}\dot{\vec{Q}} + \vec{\omega} \times \vec{Q}}$$ Q dot formula

Apply to position, velocity, acceleration

$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}$

$\vec{v}_{P/\mathcal{F}} = \vec{v}_{O'/O} + \frac{d}{dt}{}^{\mathcal{F}}\vec{r}_{P/O'} = \vec{v}_{O'/O} + \vec{v}_{P/\beta} + \vec{\omega} \times \vec{r}_{P/O'}$

$\vec{v}_{P/\beta} = {}^{\beta}\dot{\vec{r}}_{P/O}$ velocity in moving frame

For acceleration apply concept one more time, term by term

$\vec{a}_{P/\mathcal{F}} = \frac{d}{dt}\vec{v}_{P/\mathcal{F}} = {}^{\mathcal{F}}\dot{\vec{v}}_{P/\mathcal{F}} = \vec{a}_{O'/\mathcal{F}} + \frac{d}{dt}{}^{\mathcal{F}}(\vec{v}_{P/\beta}) + \frac{d}{dt}{}^{\mathcal{F}}(\vec{\omega} \times \vec{r}_{P/O'})$

$= \vec{a}_{O'/\mathcal{F}} + \vec{a}_{P/\beta} + \vec{\omega} \times \vec{v}_{P/\beta} + \dot{\vec{\omega}} \times \vec{r}_{P/O'} + \vec{\omega} \times (\vec{v}_{P/\beta} + \vec{\omega} \times \vec{r}_{P/O'})$

$$\vec{a}_{P/\mathcal{F}} = \underbrace{\vec{a}_{O'/\mathcal{F}}}_{\text{accel of } O' \text{ wrt } \mathcal{F}} + \underbrace{\vec{a}_{P/\beta}}_{\text{accel of } P \text{ wrt } O'} + \underbrace{2(\vec{\omega} \times \vec{v}_{P/\beta})}_{\text{coriolis term}} + \underbrace{\dot{\vec{\omega}} \times \vec{r}_{P/O'} + \vec{\omega} \times \vec{\omega} \times \vec{r}_{P/O'}}_{}$$

r' goes in circles around O' (angular acceleration + centripetal term)

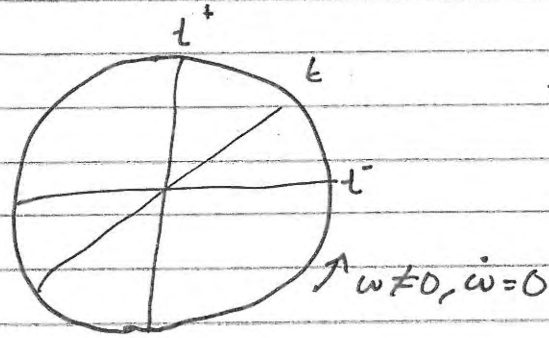
Coriolis Effect

Consider turn table

$O = O'$

$\vec{a}_{O'/O} = 0$

$\vec{r}_{P'/O'} = 0$



- bug walk on line at const speed
- look when p at p' at O'

$v_0 \rightarrow$ $\uparrow v_t$ $\begin{cases} \text{change in velocity = acceleration} \\ \text{also tangential velocity changes = acceleration} \end{cases}$

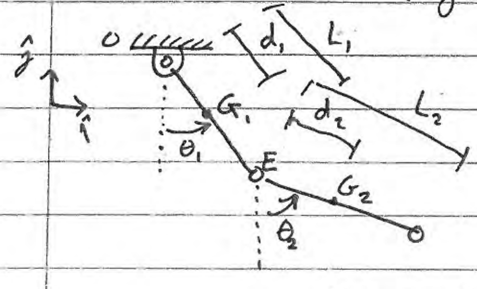
$\rightarrow$ factor of 2 in coriolis

12/2/13

TODAY

① Lag Eqs in MATLAB

[② DAEs for linkages]



$m_1, \underline{I}_1 = \underline{I}_1^G$

$m_2, \underline{I}_2 = \underline{I}_2^G$

where $\theta_2 - \theta_1 = \phi$

Lagrange Eqs (Conservative Holonomic)

$$\mathcal{L} = E_K - E_p = \text{"}T - V\text{"}$$

$\hookrightarrow E_p = E_p(q_1, q_2, \cdots)$

$\hookrightarrow E_K = E_K(q_1, q_2, \cdots, \dot{q}_1, \dot{q}_2, \cdots)$

LE:

$$\frac{\partial \mathcal{L}}{\partial q_1} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial q_2} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = 0$$

$\frac{\partial \mathcal{L}}{\partial \dot{q}_1} \Rightarrow$ treat $\dot{q}_1$ as a variable

$F(q_1, q_2, \cdots \dot{q}_1, \dot{q}_2 \cdots)$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \frac{d}{dt}F = \frac{\partial F}{\partial q_1}\dot{q}_1 + \frac{\partial F}{\partial q_2}\dot{q}_2 + \cdots \frac{\partial F}{\partial \dot{q}_1}\ddot{q}_1 + \frac{\partial F}{\partial \dot{q}_2}\ddot{q}_2 \cdots$$

ASIDE: MATLAB SYMBOLIC

A = jacobian(a, b)

b: list of variables

a: list of expressions (rows or cols)

output A = matrix

each element $\boxed{A_{ij} = \frac{\partial a_i}{\partial b_j}}$

TODAY

12/4/13

① DAEs

② Rolling

DAEs: Recall: $\vec{F} = m\vec{a}$

easy for 1 particle collection w/known interaction forces

"known": $\vec{F} = \vec{F}$ (positions and vels)

and not acceleration

Difficulty in dynamics: constraints

How to solve?

a) naive (DAE)

b) advanced (finesse the problem)

Naive: For each part: $\vec{F} = m\vec{a}$

include const. forces

Const. eqs: differentiate twice

$\Rightarrow$ restrictions on $\ddot{x}_3, \ddot{y}_3$, etc

Solve set simultaneously at every inst in time

Advanced: ⓐ Treat collection of rigidly connected particles as a rigid ~~&~~ object

$\Rightarrow \vec{M}_{/G} = I^G \alpha \hat{k}$ & LMB

(internal forces have no net torque)

ⓑ Use judicious dot products & cross products to elim const forces

What happens if we have a collection of rigid objects?

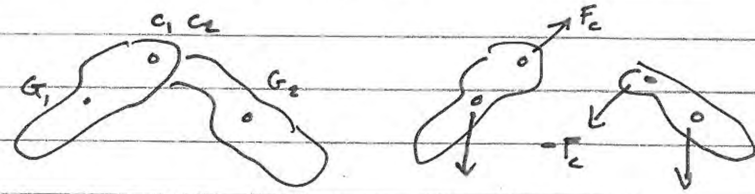
If forces known:

$$\left.\begin{aligned} M_{/G} &= I^G \alpha \\ \sum F_x &= m\ddot{x} \\ \sum F_y &= m\ddot{y} \end{aligned}\right] \quad \text{3 eqs for each rigid object (2D)}$$

I) If forces are "known"

$$F_i = F_i(\text{pos}, \text{angles}, \text{vel}, \text{rates})$$

II) What about constraints?



<u>LMB & AMB</u> $\Rightarrow$ 6 scalar eqs

unknowns: $\ddot{x}_1, \ddot{x}_2, \ddot{y}_1, \ddot{y}_2, \ddot{\theta}_1, \ddot{\theta}_2$

$F_{cx}, F_{cy} \Rightarrow$ 8 unknowns

<u>Need two more eqns</u>

$\vec{a}_{c1} = \vec{a}_{c2} \Rightarrow$ 2 eqns.

$\hookrightarrow \vec{a}_{G2} + \vec{a}_{c2/G2} = \ddot{x}_{G2}\hat{\imath} + \ddot{y}_{G2}\hat{\jmath} - \theta_2 r_{c2/G2} + \ddot{\theta}\hat{k} \times r_{c2/G2}$

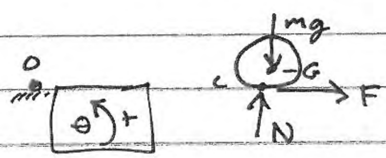
$\hookrightarrow \vec{a}_{G1} + \vec{a}_{c1/G1} = \ddot{x}_{G1}\hat{\imath} + \ddot{y}_{G1}\hat{\jmath} - \ddot{\theta}_1 r_{c1/G1} + \ddot{\theta}\hat{k} \times r_{c1/G1}$

Can handle any number of rigid objects connected by hinges

<u>ASIDE</u> Freudenstein's formulas

Disk on table

FBD



$\{LMB\} \cdot \hat{j} \Rightarrow N = mg$

$\{AMB\}_{/O}\, dt \Rightarrow \vec{H}_{/O}(t) = \vec{H}_{/O}(0) = 0\hat{k} \Rightarrow -mr\dot{x}_G \hat{k} + I^G \dot{\theta} \hat{k} = 0$

at end of exp:

$$-mr\dot{x}_G + I^G\dot{\theta} = 0 \quad ①$$

Rolling constraint applies at end: $\boxed{\vec{v}_c = \vec{0}}$

$$\Rightarrow \dot{x}_G + r\dot{\theta} = 0 \quad ②$$

① & ② Linear Eqns (homog & ind) in $\dot{x}$ & $\dot{\theta}$ $\Rightarrow \boxed{\dot{x} = 0,\ \dot{\theta} = 0}$

12/6/13

TODAY

① Generalized forces Q_i

③ 3 Advertisements

Problem: Not all forces are conservative

ex: applied, viscous, constraint

Soln: $-\frac{\partial \mathcal{L}}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = Q_i$

Q_i is generalized force associated w/ q_i

How to find Q_i?

Key idea: $\overbrace{dW = dW}^{\text{increments in work}}$

$$\sum \vec{F}^{\,i}_{applied} \cdot \underbrace{d\vec{r}_i} = \sum Q_i \, dq_i$$

$$\Rightarrow \sum \vec{F}^{\,i} \cdot \underbrace{d\vec{r}_i}_{\text{motion associated with small change in only } q_7} = Q_7 \, dq_7$$

$\cancel{Q_i \, dq_i = \frac{\partial x_i}{\partial q_i} dq_i}$

x's & y's

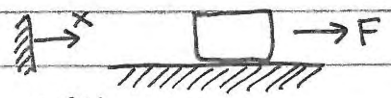
$$Q_i \, dq_i = \sum_j F_j \frac{\partial x_j}{\partial q_i} dq_i$$

all x_j where forces apply

"a" jacobian

$$J = \frac{\partial x_i}{\partial q_i}$$

ex) 1D mass



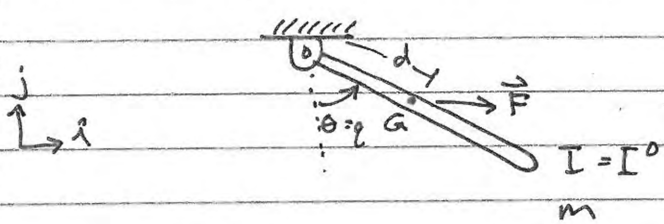
$q = x$

$$\mathcal{L} = E_k - E_p = \text{"}T - V\text{"}$$
$$= \frac{1}{2} m \dot{q}^2$$

L.E.: $\frac{-\partial \mathcal{L}}{\partial q} + \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}$

$$m\ddot{q} = Q$$
$$dW = dW$$
$$Q\,dq = F\,dx$$
$$Q = F$$
$$\boxed{m\ddot{x} = F}$$

ex) Pendulum



$$\mathcal{L} = \frac{1}{2} I \dot{\theta}^2$$

L.E.: $\frac{-\partial \mathcal{L}}{\partial \theta} + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = Q_\theta$

$$I\ddot{\theta} = Q_\theta$$

$$Q_\theta\, d\theta = \vec{F} \cdot (d\vec{r}_G)$$
$$= \vec{F} \cdot \underbrace{\frac{\partial \vec{r}_G}{\partial \theta}}_{J} d\theta$$
$$= \vec{F} \cdot \frac{d}{d\theta}\underbrace{\left(-d\cos\theta\,\hat{i} - d\sin\theta\,\hat{j}\right)}_{\hat{k}\,d\theta \times \vec{r}_G} d\theta$$

$$Q_\theta \, d\theta = \vec{F} \cdot (\hat{k}\, d\theta) \times \vec{r}_{G/O}$$

⇩ mixed triple product craziness

$$= \left(\vec{r}_{G/O} \times \vec{F}\right) \cdot \hat{k}\, d\theta$$

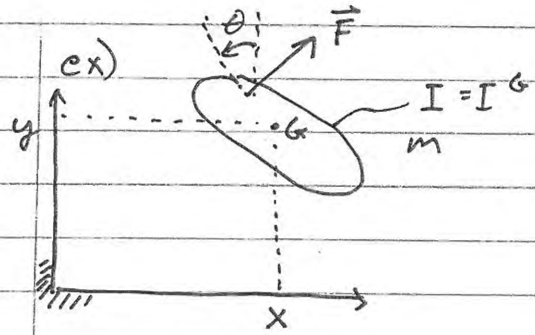
$$\Rightarrow I\ddot{\theta} = \vec{r}_{G/O} \times \vec{F} \cdot \hat{k}$$

$$= M_{/O}$$

└ ex) $M_{/O} = -mgd\sin\theta$

---

ex)



$$q = [x \quad y \quad \theta] = [q_1 \quad q_2 \quad q_3]$$

$$\mathcal{L} = \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}m\dot{q}_2^2 + \frac{1}{2}I\dot{q}_3^2$$

L.E.: x: $-\frac{\partial \mathcal{L}}{\partial x} + \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = Q_x$

$$m\ddot{x} = Q_x$$

y: $m\ddot{y} = Q_y$

θ: $I\ddot{\theta} = Q_\theta$

$$Q_x dx = \vec{F} \cdot \underbrace{d\vec{r}_c}_{\text{when only } dx \neq 0}$$

$$= F_x dx$$

$$Q_x = F_x$$

also $Q_y = F_y$

$$Q_\theta \, d\theta = \vec{F} \cdot \underbrace{\vec{r}_c}_{\text{when only } d\theta \neq 0}$$

$$= \vec{F} \cdot \frac{\partial \vec{r}_c}{\partial \theta} d\theta$$

$$\boxed{Q_\theta = \vec{F} \cdot \frac{\partial \vec{r}_c}{\partial \theta}}$$ → 3^{rd} column of $J_{2\times3}$ dotted w/ $\vec{F}$

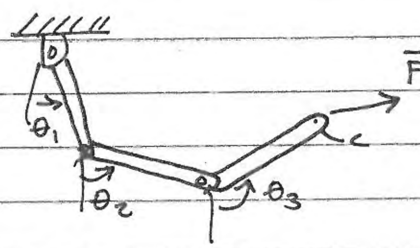
$$Q_\theta = (\vec{r}_{c/G} \times \vec{F}) \cdot \hat{k}$$

Can supplement eqns w/ kinematic constraint eqs:

e.g. $\ddot{\vec{r}}_c = \vec{0}$

Solve constrained eqns

⇒ set of DAEs

ex) 

Calc q_{θ_1} q_{θ_2} q_{θ_3}

look at work done by changing each θ

$\vec{a}_c = \vec{0}$

3 diff eqn + 2 constraint eqns